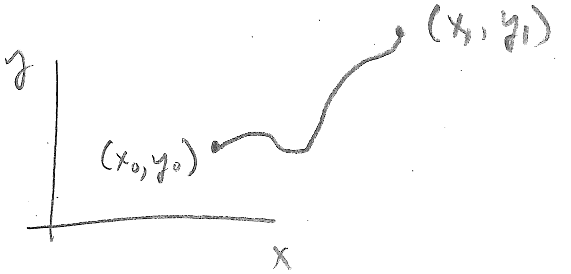
Shortest distance between 2 pts on a plane is a straight line



Length of an arbitrary path $L = \int ds = \sqrt{dx^2 + dy^2}$

Choose x as independent variable $L = \int dx \underbrace{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}_{f(y, y', x)}$

First form of Euler: $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$

Since f indep. of y, $\frac{\partial f}{\partial y'} = \text{const}$

$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \Rightarrow y' = \text{const}, C \Rightarrow y = Cx + D$

use endpts to fix constants: $y = y_0 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$

Alternatively, second form of Euler: $\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'} - f\right) - \frac{\partial f}{\partial x} = 0$

Since f indep of x, $\underbrace{y' \frac{\partial f}{\partial y'} - f}_{\underbrace{\frac{y'^2}{\sqrt{1+y'^2}} - \sqrt{1+y'^2}}_{\frac{-1}{\sqrt{1+y'^2}}}} = \text{const}$

$\frac{-1}{\sqrt{1+y'^2}} = \text{const} \Rightarrow y' = \text{const}$

(same result as before)

Alt. choose a parameter p

$$L = \int dp \underbrace{\sqrt{\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2}}_{f}$$

f indep of x or y $\Rightarrow \begin{cases} \dfrac{\partial f}{\partial \dot{x}} = \dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = A \\ \dfrac{\partial f}{\partial \dot{y}} = \dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = B \end{cases}$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{B}{A} \quad \Rightarrow \text{ straight line.}$$

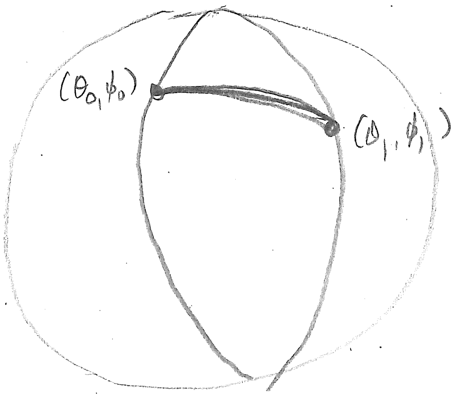
2nd form of euler: f independent of p so

$$\dot{x}\frac{\partial f}{\partial \dot{x}} + \dot{y}\frac{\partial f}{\partial \dot{y}} - f = \frac{\dot{x}^2}{\sqrt{\dot{x}^2+\dot{y}^2}} + \frac{\dot{y}^2}{\sqrt{\dot{x}^2+\dot{y}^2}} - \sqrt{\dot{x}^2+\dot{y}^2} = 0$$

so this does not yield anything useful

shortest distance between 2 pts on a sphere is a great circle

cf Marion (3e) p 181



$$L = \int ds = \int \sqrt{(R\,d\theta)^2 + (R\sin\theta\, d\phi)^2}$$

choose ϕ as independent variable

$$L = R \int d\phi \underbrace{\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}}_{f(\theta,\, \theta',\, \phi)}$$

Use 2nd form of Euler:

$$\frac{d}{d\phi}\left(\theta' \frac{\partial f}{\partial \theta'} - f \right) - \frac{\partial f}{\partial \phi} = 0 \qquad \text{where } \theta' = \frac{d\theta}{d\phi}$$

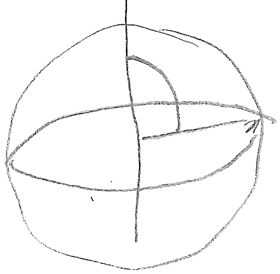
$$\frac{\partial f}{\partial \theta'} = \frac{\theta'}{\sqrt{\theta'^2 + \sin^2\theta}} \quad \Rightarrow \quad \theta' \frac{\partial f}{\partial \theta'} - f = \frac{-\sin^2\theta}{\sqrt{\theta'^2 + \sin^2\theta}}$$

Since f is indep of ϕ, $\quad \dfrac{\sin^2\theta}{\sqrt{\theta'^2 + \sin^2\theta}} = C$

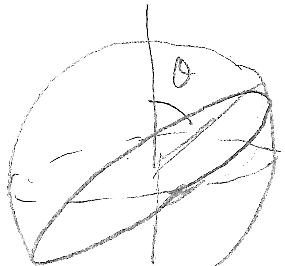
Solve for θ' $\Rightarrow$ $\quad \dfrac{d\theta}{d\phi} = \dfrac{\sin\theta}{C}\sqrt{\sin^2\theta - C^2}$

Before solving this eqn, let's think about what the solution should look like

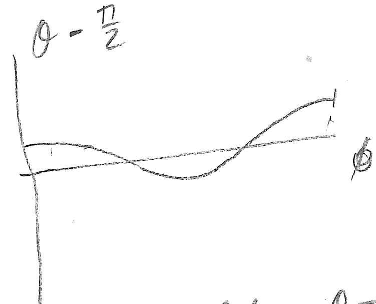
Great circles



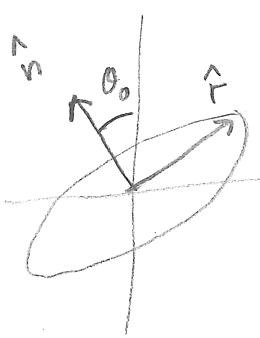
equator $\Rightarrow$ $\theta = \frac{\pi}{2}$



$\Rightarrow$



Looks approximately like $\theta - \frac{\pi}{2} \sim \cos\phi$



Points on great circle satisfy

$$\hat{r}^2 = 1$$

$\hat{r}\cdot\hat{n} = 0$ for $\hat{n}$ perpendicular to circle

$$\hat{n} = (\cos\theta_0,\ \sin\theta_0\cos\phi_0,\ \sin\theta_0\sin\phi_0)$$

$$\hat{r} = (\cos\theta,\ \sin\theta\cos\phi,\ \sin\theta\cos\phi)$$

$$\hat{r}\cdot\hat{n} = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\,(\underbrace{\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0}_{\cos(\phi-\phi_0)}) = 0$$

$$\Rightarrow \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta_0}{\cos\theta_0}\cos(\phi-\phi_0) = 0$$

ie $\boxed{\cot\theta = -\tan\theta_0\cos(\phi-\phi_0)}$

NB $\cot\theta \sim \frac{\pi}{2} - \theta$ near $\theta = \frac{\pi}{2}$

$$\frac{d\theta}{d\phi} = \frac{\sin\theta}{C}\sqrt{\sin^2\theta - C^2}$$

$\sin^2\theta + \cos^2\theta = 1$

$1 + \cot^2\theta = \csc^2\theta$

$$d\phi = \frac{C\,d\theta}{\sin\theta\sqrt{\sin^2\theta - C^2}}$$

$$= \frac{C\,d\theta}{\sin^2\theta\sqrt{1 - C^2\csc^2\theta}}$$

$$= \frac{C\,\csc^2\theta\,d\theta}{\sqrt{1 - C^2(1+\cot^2\theta)}}$$

$$= \frac{C}{\sqrt{1-C^2}}\frac{\csc^2\theta\,d\theta}{\sqrt{1 - \left(\frac{C^2}{1-C^2}\right)\cot^2\theta}}$$

$$= \frac{A\,\csc^2\theta\,d\theta}{\sqrt{1-(A\cot\theta)^2}}$$

Let $\xi = A\cot\theta$

$d\xi = -A\csc^2\theta\,d\theta$

$$d\phi = \frac{-d\xi}{\sqrt{1-\xi^2}}$$

$\xi = \cos\chi$

$d\xi = -\sin\chi\,d\chi$

$$d\phi = \frac{\sin\chi\,d\chi}{\sqrt{1-\cos^2\chi}}$$

$$d\phi = d\chi$$

$\Rightarrow \quad \phi = \chi + \phi_0$

$\chi = \phi - \phi_0$

$\xi = \cos\chi = \boxed{\cos(\phi-\phi_0) = A\cot\theta}$ ← great circle if $A = -\cot\theta_0$

Try 1st form of Euler

$$L = \int ds = R \int \sqrt{d\theta^2 + \sin^2\theta \, d\phi^2}$$

$$= R \int d\phi \underbrace{\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}}_{f}$$

$$\frac{\partial f}{\partial \theta} = \frac{\sin\theta\cos\theta}{\sqrt{\dot\theta^2 + \sin^2\theta}}$$

$$\frac{\partial f}{\partial \dot\theta} = \frac{\dot\theta}{\sqrt{\dot\theta^2 + \sin^2\theta}}$$

$$0 = \frac{d}{d\phi}\left(\frac{\partial f}{\partial \dot\theta}\right) - \frac{\partial f}{\partial \theta}$$

$$= \frac{\ddot\theta}{\sqrt{\quad}} - \frac{\dot\theta(\dot\theta\ddot\theta + \sin\theta\cos\theta\,\dot\theta)}{(\quad)^{3/2}} - \frac{\sin\theta\cos\theta}{\sqrt{\quad}}$$

$$0 = (\dot\theta^2 + \sin^2\theta)\ddot\theta - \dot\theta^2\ddot\theta - \dot\theta^2\sin\theta\cos\theta - (\dot\theta^2 + \sin^2\theta)\sin\theta\cos\theta$$

$$= \sin^2\theta\,\ddot\theta - 2\dot\theta^2\sin\theta\cos\theta - \sin^3\theta\cos\theta$$

$$0 = \ddot\theta - 2\cot\theta\,\dot\theta^2 - \sin\theta\cos\theta$$

[other approach to great circle not so easy]

$$L = \int ds = R \int dp \underbrace{\sqrt{\left(\frac{d\theta}{dp}\right)^2 + \sin^2\theta \left(\frac{d\phi}{dp}\right)^2}}_{f}$$

2nd form of euler yields zero.
1st form of euler.

$$\frac{\partial f}{\partial \dot\phi} = \frac{\sin^2\theta\, \dot\phi}{\sqrt{\dot\theta^2 + \sin^2\theta\, \dot\phi^2}} = c$$

$$\sin^4\theta\, \dot\phi^2 = c^2 \left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2 \right)$$

$$\dot\theta^2 = \left(\frac{\sin^2\theta}{c^2} - 1 \right) \sin^2\theta\, \dot\phi^2$$

$$\frac{\partial f}{\partial \theta} = \frac{\sin\theta \cos\theta\, \dot\phi^2}{\sqrt{\dot\theta^2 + \sin^2\theta\, \dot\phi^2}}$$

$$\frac{\partial f}{\partial \dot\theta} = \frac{\dot\theta}{\sqrt{\dot\theta^2 + \sin^2\theta\, \dot\phi^2}}$$

$$\frac{d}{dp}\left[\frac{\dot\theta}{\sqrt{\dot\theta^2 + \sin^2\theta\, \dot\phi^2}} \right] = \frac{\sin\theta \cos\theta\, \dot\phi^2}{\sqrt{\dot\theta^2 + \sin^2\theta\, \dot\phi^2}}$$

Remember we have an arbitrary parameter p here.
we could choose $p = \phi$ in which case $\dot\theta^2$ eqn
reduces to previous