Calculus of variations

[Feynman vol II, ch 19]

Consider a function $f(x)$

What happens to f if we vary independent variable x?

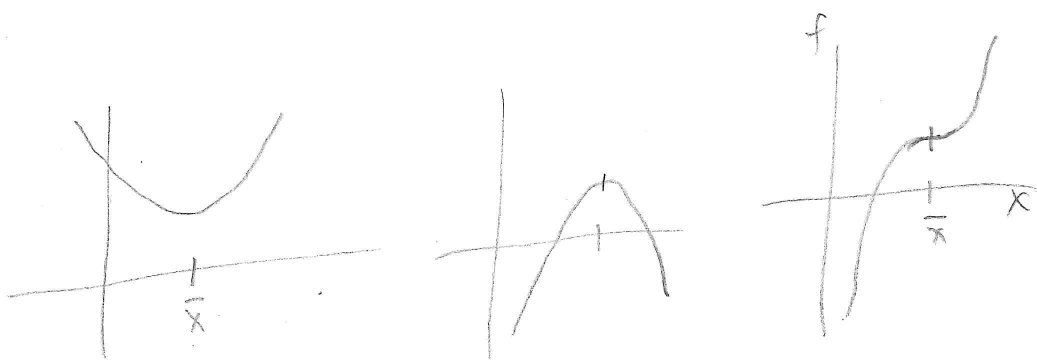
Derivative $\frac{df}{dx}$

If f doesn't change at $\bar{x}$

$$\left.\frac{df}{dx}\right|_{x=\bar{x}} = 0$$

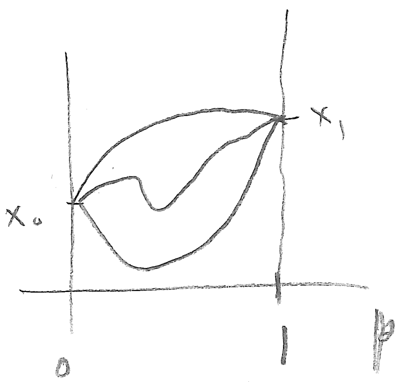
then $\bar{x}$ is a stationary point of f

[$\underbrace{\text{max/min}}_{\text{extremum}}$/ pt of inflection]



Now consider a family of continuous functions $x(p)$ ← parameter

w/ fixed endpoints: $\begin{cases} x_0 = x(0) \\ x_1 = x(1) \end{cases}$



paths from x_0 to x_1

Let $J[x(p)]$ be a function of these paths

We call this a "functional" [square brackets]

What happens to J if we vary the path $x(p)$?

Functional derivative $\frac{\delta J}{\delta x}$

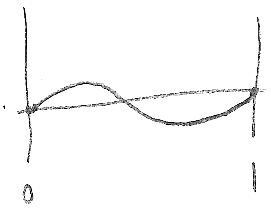
Consider some particular path $\bar{x}(p)$

If J doesn't change for paths near $\bar{x}(p)$

$$\left.\frac{\delta J}{\delta x}\right|_{x(p)=\bar{x}(p)} = 0$$

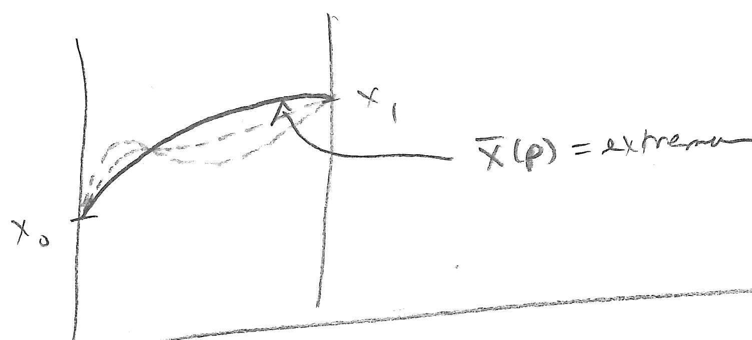
then $\bar{x}(p)$ is a stationary pt. of J.

Let $\eta(p)$ be any arbitrary function that vanishes at endpoints $\begin{cases} \eta(0) = 0 \\ \eta(1) = 0 \end{cases}$



Define a one-parameter family of paths near $\bar{x}(p)$

$$x_\epsilon(p) = \bar{x}(p) + \epsilon\,\eta(p)$$

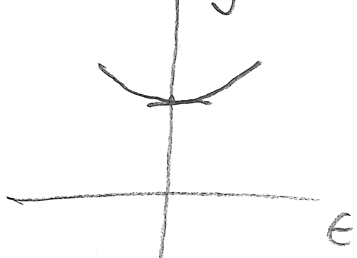


observe $\begin{cases} x_\epsilon(0) = x_0 \\ x_\epsilon(1) = x_1 \end{cases}$

[So these are indeed all paths from x_0 to x_1]

If $\bar{x}(p)$ is a minimum of J, then

in particular $J[x_\epsilon(p)] \geq J[\bar{x}(p)]$



$$\Rightarrow \quad \left.\frac{dJ[x_\epsilon(p)]}{d\epsilon}\right|_{\epsilon=0} = 0$$

for *any* choice of $\eta(p)$

More generally, this is true if $\bar{x}(p)$ is a stationary point of J [could be saddle point]

Suppose $J[x(p)]$ is given by the integral

$$\int_0^1 dp\ f(x(p), x'(p), p) \qquad \text{where } x' = \frac{dx}{dp}$$

then

$$J[x_\epsilon(p)] = \int_0^1 dp\ f(\bar{x} + \epsilon\eta, \bar{x}' + \epsilon\eta', p)$$

Use chain rule to take derivative.

$$\frac{dJ}{d\epsilon} = \int_0^1 dp\left[\frac{\partial f}{\partial x}\frac{d(\bar{x}+\epsilon\eta)}{d\epsilon} + \frac{\partial f}{\partial x'}\frac{d(\bar{x}'+\epsilon\eta')}{d\epsilon}\right]$$

$$= \int_0^1 dp\left[\frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial x'}\eta'\right]$$

Recall $\frac{\partial f}{\partial x'}\frac{d\eta}{dp} = \frac{d}{dp}\left(\frac{\partial f}{\partial x'}\eta\right) - \frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right)\eta$

$$\frac{dJ}{d\epsilon} = \int_0^1 dp\left[\frac{\partial f}{\partial x} - \frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right)\right]\eta + \underbrace{\int_0^1 dp\,\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\eta\right)}_{\underbrace{\left.\frac{\partial f}{\partial x'}\eta(p)\right|_0^1}_{\text{vanishes because } \eta(0)=\eta(1)=0}}$$

$$\frac{dJ}{d\epsilon} = \int_0^1 dp\left[\frac{\partial f}{\partial x} - \frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right)\right]\eta$$

If $\bar{x}(p)$ is a stationary point of J

then $\frac{dJ}{d\epsilon} = 0$ for any $\eta(p)$.

This can only occur if

$$\boxed{\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right) - \frac{\partial f}{\partial x} = 0}$$

Euler's eqn (1744)

Corollary: if f is independent of x
(ie it depends only on x') then

$$\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right) = 0$$

ie $\frac{\partial f}{\partial x'}$ = constant (indep of p)

Second form of Euler's eqn

Given $f(x, x', p)$

Consider
$$\frac{df}{dp} = \underbrace{\frac{\partial f}{\partial x}}_{\substack{\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right) \\ \text{by Euler (first form)}}} \frac{dx}{dp} + \frac{\partial f}{\partial x'}\frac{dx'}{dp} + \frac{\partial f}{\partial p}$$

$$= \frac{d}{dp}\left(x' \frac{\partial f}{\partial x'}\right) + \frac{\partial f}{\partial p}$$

$$\boxed{0 = \frac{d}{dp}\left(x' \frac{\partial f}{\partial x'} - f\right) + \frac{\partial f}{\partial p}}$$

2nd form of Euler's eqn

Corollary 2: if f has no explicit dependence on p

i.e. $\frac{\partial f}{\partial p} = 0$ then

$$\frac{d}{dp}\left(x' \frac{\partial f}{\partial x'} - f\right) = 0$$

i.e. $x' \frac{\partial f}{\partial x'} - f = \text{const}$ (indep of p)

[Wait until class after introducing prev. material & students can digest it a bit]

Let's introduce a convenient shorthand for previous calculation ("δ- notation").

δx = variation of the path (what we called $\epsilon\eta(p)$ earlier)

$$J[x] = \int dp\, f(x, x', p)$$

$$\delta J = \int dp \left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial x'}\delta x' \right]$$

$$\text{where } \delta x' = \delta\left(\frac{dx}{dp}\right) = \frac{d}{dp}\delta x$$

$$\therefore \quad \frac{\partial f}{\partial x'}\delta x' = \frac{d}{dp}\left(\frac{\partial f}{\partial x'}\delta x\right) - \frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right)\delta x$$

$$\delta J = \int dp \left[\frac{\partial f}{\partial x} - \frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right)\right]\delta x + \underbrace{\int dp\, \frac{d}{dp}\left(\frac{\partial f}{\partial x'}\delta x\right)}_{\underbrace{\frac{\partial f}{\partial x'}\delta x\Big|_0^1}_{0}}$$

because δx vanishes at endpoints

$\delta J = 0$ for any δx iff

$$\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right) - \frac{\partial f}{\partial x} = 0$$

$$J = \int dp\, f(x, x', p)$$

$$\delta J = 0 \quad \Rightarrow \quad \begin{cases} \dfrac{d}{dp}\left(\dfrac{\partial f}{\partial x'}\right) - \dfrac{\partial f}{\partial x} = 0 \\[2ex] \dfrac{d}{dp}\left(x'\dfrac{\partial f}{\partial x'} - f\right) + \dfrac{\partial f}{\partial p} = 0 \end{cases}$$

Two important corollaries

1) __if__ f is independent of x (ie $\frac{\partial f}{\partial x} = 0$)

then $\frac{d}{dp}\left(\frac{\partial f}{\partial x'}\right) = 0 \Rightarrow \boxed{\frac{\partial f}{\partial x'} = \text{const}}$

2) if no explicit dependence of f on p (ie $\frac{\partial f}{\partial p} = 0$)

then $\frac{d}{dp}\left(x' \frac{\partial f}{\partial x'} - f\right) = 0$ ie $\boxed{x' \frac{\partial f}{\partial x'} - f = \text{const}}$

(mom cons,
energy cons.)