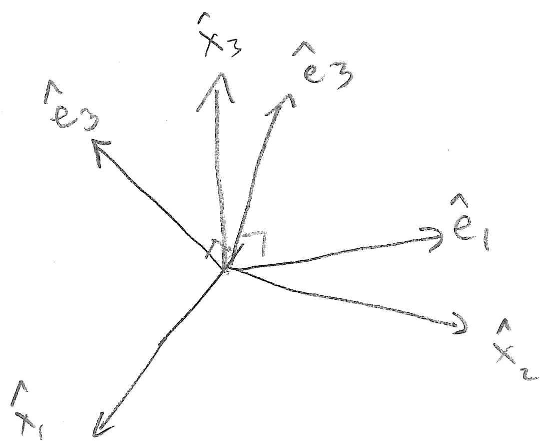
To understand rotational motion in the space-fixed frame, we need a set of coordinates that describe the orientation of a rigid body relative to the space-fixed frame

$\Rightarrow$ Euler angles

How many coordinates are necessary?



Rotation matrix $R_{ki} = e^k{}_i$

takes you from $\hat{x}_i$ to $\hat{e}^k$

How many independent parameters in R?

$e^k{}_i$ is 9 parameters

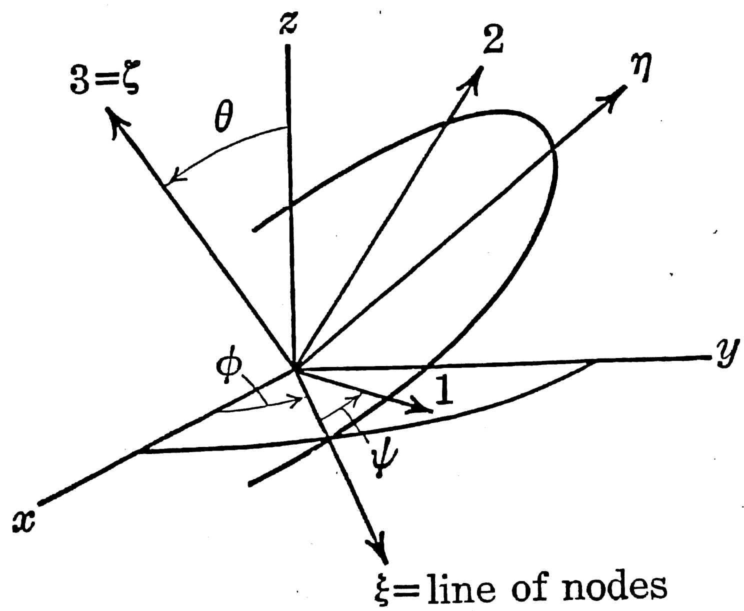
unit $(\hat{e}^1)^2 = (\hat{e}^2)^2 = (\hat{e}^3)^2 = 1$

orthogonal $\hat{e}^1 \cdot \hat{e}^2 = \hat{e}^2 \cdot \hat{e}^3 = \hat{e}^3 \cdot \hat{e}^1 = 0$

$\Big\}$ 6 conditions

$\Rightarrow$ 3 free parameters required to describe orientation

We'll use the Euler angles: ϕ, θ, ψ

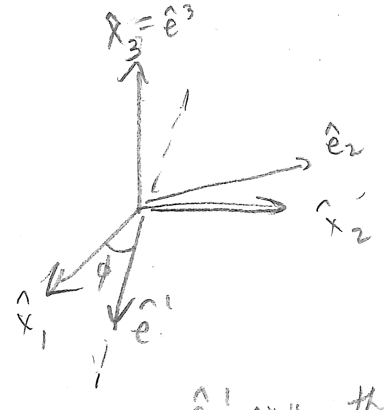


z
2
η
$3=\zeta$
θ
ϕ
y
1
ψ
x
ξ=line of nodes

v2

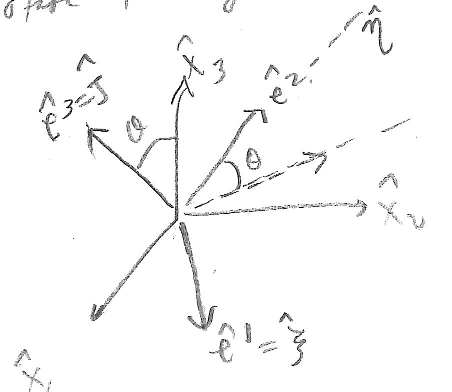
Start w/ $\hat{x}_i$ and $\hat{e}^i$ aligned.

① rotate through ϕ about the $\hat{x}_3 = \hat{e}^3$ axis



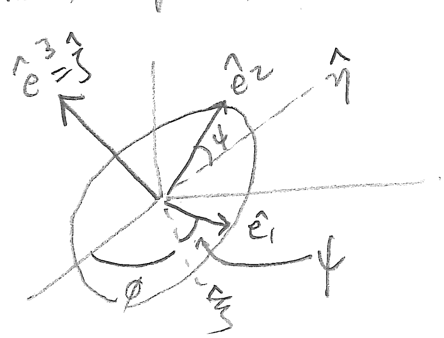
Call the new $\hat{e}^1$ axis the line of nodes $\hat{\xi}$

② rotate through θ about the line of nodes



Call the new $\hat{e}^2$ axis η, and new $\hat{e}^3$ axis ζ

③ Rotate through ψ about the $\hat{e}^3$ axis



[hand out euler angle]

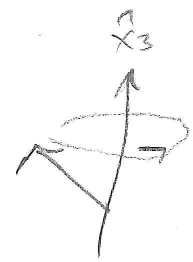
~~Call $\hat{e}^3$ axis~~

How do we describe the motion when Euler angles change?

Consider a figure of revolution about $\hat{e}^3$ axis ($\Rightarrow I_1 = I_2$)

(eg a disk)

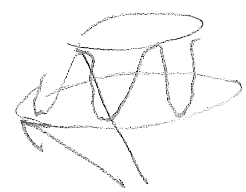
$\dot{\psi} \neq 0 \Rightarrow$ spinning about $\hat{\zeta} = \hat{e}^3$



$\dot{\phi} \neq 0 \Rightarrow \hat{e}^3$ axis circles the $\hat{x}_3$ axis

precession

$\dot{\theta} \neq 0 \Rightarrow$ angle of tip is changing



if $\dot{\theta}$ periodic,

called nutation (nodding)

Let's write $\vec{\omega}$ in terms of Euler angles

From figure

$$\vec{\omega} = \underbrace{\dot{\psi}\,\hat{e}^3}_{\text{spin rate}} + \underbrace{\dot{\phi}\,\hat{X}_3}_{\text{precession rate}} + \underbrace{\dot{\theta}\,\hat{\xi}}_{\text{"tilting" rate}}$$

not orthogonal!

$$\hat{X}_3 = \hat{e}^3 \cos\theta + \hat{\eta}\sin\theta$$

$$\vec{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}^3 + \dot{\phi}\sin\theta\,\hat{\eta} + \dot{\theta}\,\hat{\xi}$$

$$\hat{\eta} = \hat{e}^1 \sin\psi + \hat{e}^2\cos\psi$$

$$\hat{\xi} = \hat{e}^1 \cos\psi - \hat{e}^2\sin\psi$$

$$\vec{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{e}^3 + (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\,\hat{e}^1$$

$$+ (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\,\hat{e}^2$$

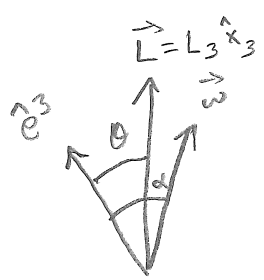
$$\boxed{\begin{aligned} \omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta \end{aligned}}$$

← NB. ω_i here are components in the body fixed frame (should have primes but we continue to omit them)

Recall: in space-fixed frame $\vec{L} = \text{const}$ if $\vec{\tau}^{ext} = 0$

For convenience choose $\vec{L}$ along $\hat{x}_3$ axis.

For a symmetric object, $\hat{e}^3$ and $\vec{\omega}$ rotate around $\vec{L}$



$|\vec{\omega}|, \theta, \alpha$ are all constant in time

$\omega_3 = \omega \cos\alpha$

$\omega_\perp = \omega \sin\alpha \implies \omega_\perp = \omega_3 \tan\alpha$

Also $I_3 \tan\theta = I_\perp \tan\alpha$

Since $\dot{\theta} = 0$,

$\omega_1 = \dot{\phi} \sin\theta \sin\psi$

$\omega_2 = \dot{\phi} \sin\theta \cos\psi$

$\omega_3 = \dot{\psi} + \dot{\phi} \cos\theta$

$$\implies \omega_\perp = \sqrt{\omega_1^2 + \omega_2^2} = \dot{\phi} \sin\theta = \omega_3 \tan\alpha$$

$$\implies \dot{\phi} = \omega_3 \frac{\tan\alpha}{\sin\theta} = \text{const}$$

$$= \frac{\omega_3}{\sin\theta}\left(\frac{I_3 \tan\theta}{I_\perp}\right) = \frac{I_3}{I_\perp}\frac{\omega_3}{\cos\theta}$$

$$= \frac{\omega_3}{I_\perp} I_3 \sec\theta = \frac{\omega_3}{I_\perp} I_3 \sqrt{1+\tan^2\theta} = \frac{\omega_3}{I_\perp}\sqrt{I_3^2 + (I_3\tan\theta)^2}$$

$$\dot{\phi} = \frac{\omega_3}{I_\perp}\sqrt{I_3^2 + I_\perp^2 \tan^2\alpha}$$

W2

Calculate $\dot{\psi}$:

$$\dot{\psi} = \omega_3 - \dot{\phi}\cos\theta$$

$$= \omega_3 - \frac{I_3}{I_\perp}\omega_3$$

$$= \frac{I_\perp - I_3}{I_\perp}\omega_3$$

$$\boxed{\dot{\psi} = -\Omega}$$

rate of spin

makes sense: Ω is rotation of $\vec{\omega}$ + $\vec{L}$ about body fixed axes

If $\dot{\psi} = 0 \Rightarrow$ pure precession

$\Rightarrow \vec{L}$ + $\vec{\omega}$ remain fixed in body-fixed frame

[Look at diagram]

Finally

$$\dot{\phi}\sin\theta = \omega_3 \tan\alpha \quad (*)$$

$$\dot{\phi}\cos\theta = \omega_3 - \dot{\psi} = \omega_3\left[1 - \frac{I_\perp - I_3}{I_\perp}\right] = \frac{I_3}{I_\perp}\omega_3$$

$$\dot{\phi}^2 = \left(\frac{I_3\omega_3}{I_\perp}\right)^2 + (\omega_3\tan\alpha)^2$$

$$= \frac{\omega_3^2}{I_\perp^2}\left(I_3^2 + I_\perp^2\tan^2\alpha\right)$$

omit?

sign fixed by $(*)$

$$\boxed{\dot{\phi} = \frac{\omega_3}{I_\perp}\sqrt{I_3^2 + I_\perp^2\tan^2\alpha}}$$

rate of precession

From above

$$\boxed{\frac{\dot{\phi}}{\dot{\psi}} = \frac{\sqrt{I_3^2 + I_\perp^2\tan^2\alpha}}{I_\perp - I_3}}$$

$\Rightarrow$ ~300 for earth

If $I_3 \approx I_\perp$ then rotation is mostly precession