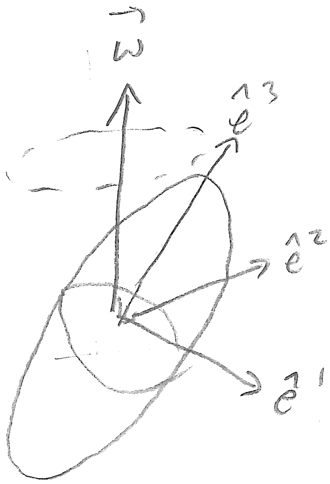
[can find instant axis by expanding $R = 1 + \epsilon$ where $\epsilon = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ & 0 & \omega_1 \\ & & 0 \end{pmatrix}$ see 1999 notes]

Suppose the rigid body is rotating about some axis $\vec{\omega}$ ($\vec{\omega}$ can change over time, so we refer to it as the instantaneous axis of rotation)




The body-fixed axes rotate as well (w.r.t. space-fixed frame)

$$\frac{d\hat{e}^k}{dt} = \vec{\omega} \times \hat{e}^k$$

[recall earlier discussion of position vectors]

Any vector that is fixed (not changing) in the body-fixed frame also rotates in space-fixed frame

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$

---

If the vector is changing in the body-fixed frame we have to include that too

$$\underset{\substack{\text{change in}\\\text{space fixed}\\\text{frame}}}{\frac{d\vec{A}}{dt}} = \underset{\substack{\text{change in}\\\text{body-fixed}\\\text{frame}}}{\left(\frac{d\vec{A}}{dt}\right)_{body}} + \underset{\substack{\text{rotation}\\\text{with respect}\\\text{to space-fixed}\\\text{frame}}}{\vec{\omega} \times \vec{A}}$$

Can do it more mathematically

$$\vec{A} = \sum A'_k \hat{e}^k$$

$$\frac{d\vec{A}}{dt} = \underbrace{\sum \frac{dA'_k}{dt} \hat{e}^k}_{\equiv \left(\frac{d\vec{A}}{dt}\right)_{body}} + \underbrace{\sum A'_k \underbrace{\frac{d\hat{e}^k}{dt}}_{\vec{\omega} \times \hat{e}^k}}_{\vec{\omega} \times \vec{A}}$$

We proved $\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\tau}$ in any IRF

$$\Rightarrow \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = \vec{\tau} \quad (*) \quad \text{[body-fixed frame not inertial]}$$

---

Let's evaluate this eqn in body-fixed coordinates

Recall $\vec{L} = \sum_k L'_k \hat{e}^k = \sum I_k \omega'_k \hat{e}^k$

$$\left(\frac{d\vec{L}}{dt}\right)_{body} = \sum I_k \frac{d\omega'_k}{dt} \hat{e}^k$$

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{e}^1 & \hat{e}^2 & \hat{e}^3 \\ \omega'_1 & \omega'_2 & \omega'_3 \\ I_1\omega'_1 & I_2\omega'_2 & I_3\omega'_3 \end{vmatrix} = \hat{e}^1 \omega'_2\omega'_3 (I_3 - I_2) + \cdots$$

$$\vec{\tau} = \sum \tau'_k \hat{e}^k$$

---

Thus the components of $(*)$ are

$$\left.\begin{aligned} I_1 \frac{d\omega'_1}{dt} + (I_3 - I_2)\,\omega'_2\omega'_3 &= \tau'_1 \\ I_2 \frac{d\omega'_2}{dt} + (I_1 - I_3)\,\omega'_1\omega'_3 &= \tau'_2 \\ I_3 \frac{d\omega'_3}{dt} + (I_2 - I_1)\,\omega'_1\omega'_2 &= \tau'_3 \end{aligned}\right\}$$

Euler equations for a rigid body (in body-fixed frame)

← These are hard to use if torque $\neq 0$ because must evaluate $\vec{\tau}$ in body-fixed frame

(cyclic perms)

Torque-free motion of a rigid body (in body-fixed coords)

Drop the primes from Euler's eqns
but remember we are using body-fixed coordinates!

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$$

---

First assume that object is initially rotating about one of the principle axes, e.g. 3, so $\omega_1 = \omega_2 = 0$

$$I\dot{\omega}_1 = 0$$
$$I\dot{\omega}_2 = 0$$
$$I\dot{\omega}_3 = 0$$

so $\omega_1$ and $\omega_2$ remain zero, and $\omega_3 = \text{const}$

Question: Is this motion stable?

Suppose it receives a perturbation away from the principle axis; will the perturbation remain small?

Assume object is <u>almost</u> rotating about principle axis 3

$\Rightarrow \omega_1, \omega_2 \ll \omega_3$

$$I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2 \approx 0 \quad \Rightarrow \quad \omega_3 \approx \text{const}$$

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\,\omega_3\right)\omega_2 \approx \alpha\,\omega_2 \quad \text{for const } \alpha$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\,\omega_3\right)\omega_1 \approx \beta\,\omega_1$$

coupled o.d.e

$$\ddot{\omega}_1 \approx \alpha \dot{\omega}_1 \approx \alpha\beta\,\omega_1$$
$$\ddot{\omega}_2 = \beta \dot{\omega}_1 \approx \beta\alpha\,\omega_2$$

If $\alpha\beta < 0 \Rightarrow$ harmonic osc $\Rightarrow$ small $\omega_1$ remains small
small $\omega_2$ " "

~~If also true for $\omega_2$, then~~ $\Rightarrow$ rotation about 3 is stable

If $\alpha\beta > 0 \Rightarrow$ exponential growth, $\omega_1 \approx \omega_1(0)\, e^{\sqrt{\alpha\beta}\, t}$ (at least initially)

rotation about 3 is unstable

[as $\omega_1 \uparrow$, $\dot{\omega}_3 \approx 0$ no longer valid]

Principle moments can be ordered from (~~least~~ min, intermediate, ~~most~~ max)

---

Suppose $I_3$ is max or min moment of inertia

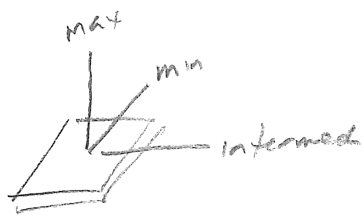
Then $$\alpha\beta = \frac{-(I_2 - I_3)(I_1 - I_3)}{I_1 I_2} < 0$$ $\Rightarrow$ stable rotation about max/min axis

---

Suppose $I_3$ is intermediate moment

eg $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$

$\alpha\beta > 0$ $\Longrightarrow$ unstable rotation about intermediate axis (intermediate axis theorem)




[Demo: book]

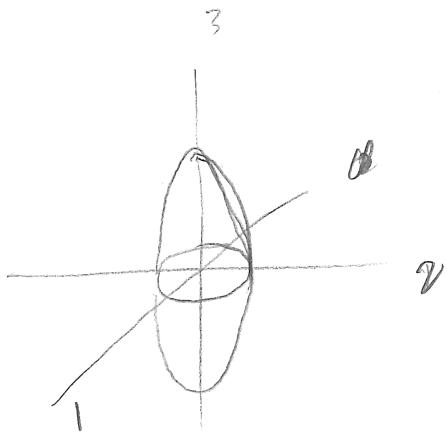
hard to catch when spun about intermediate axis

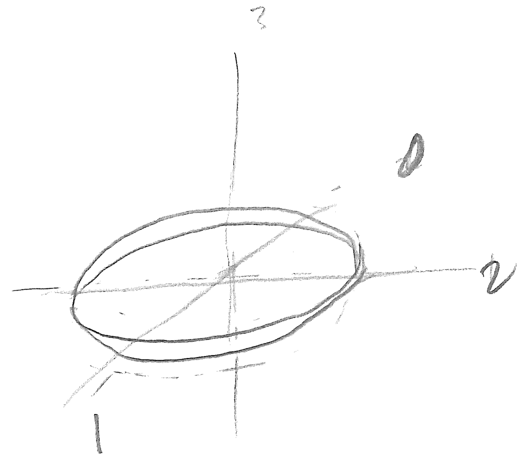
[Demo: tennis racket]

A *symmetric top* is an object w/ two equal principle moments

$$I_1 = I_2 = I_\perp \qquad\qquad I = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

eg: figure of revolution




prolate ellipsoid

$I_3 < I_\perp$

oblate ellipsoid (earth)

$I_3 > I_\perp$

$\left( \text{earth: } \dfrac{I_3 - I_\perp}{I_3} \simeq \dfrac{1}{300} \right)$

# Torque free motion of a symmetric top $(I_1 = I_2 \equiv I_\perp)$

Euler $\Rightarrow$ $\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{const}$

$$\dot{\omega}_1 = \left(\frac{I_\perp - I_3}{I_\perp}\omega_3\right)\omega_2 = -\Omega\,\omega_2 \quad \text{where} \quad \Omega = \frac{I_3 - I_\perp}{I_\perp}\omega_3 = \text{const}$$

[conventions of Fetter-Walecka, Thornton-Marion

[opposite those of Goldstein, Eck

$$\dot{\omega}_2 = \left(\frac{I_3 - I_\perp}{I_\perp}\omega_3\right)\omega_1 = \Omega\,\omega_1$$

$\Rightarrow$ $\ddot{\omega}_1 = -\Omega^2 \omega_1$

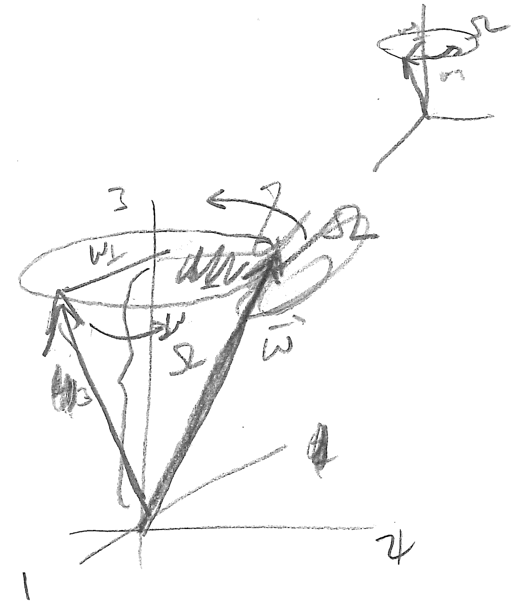
[could have $\cos(\Omega t + \varphi)$ but set $\varphi = 0$]

$\Rightarrow$ $\omega_1(t) = \omega_\perp \cos(\Omega t)$

$\dot{\omega}_1 = -\Omega\,\omega_\perp \sin(\Omega t) = -\Omega\,\omega_2$

$\Rightarrow$ $\omega_2 = \omega_\perp \sin(\Omega t)$

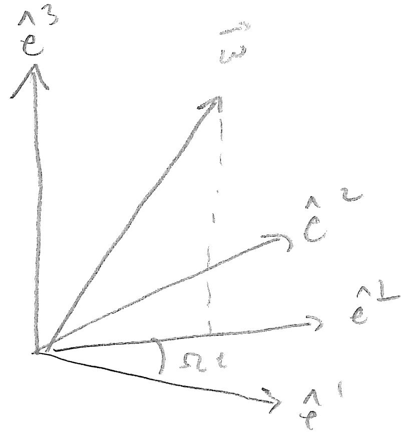
$$\vec{\omega} = \begin{pmatrix} \omega_\perp \cos(\Omega t) \\ \omega_\perp \sin(\Omega t) \\ \omega_3 \end{pmatrix} \quad \text{in body-fixed frame}$$




$\vec{\omega}$ describes a cone about $\hat{e}_3$ (if $\Omega \neq 0$)

"body cone"

if $\omega_3 = 0$ then $\omega_1, \omega_2 = \underline{\text{const}}$




Define $\hat{e}^\perp = \hat{e}^1 \cos(\Omega t) + \hat{e}^2 \sin(\Omega t)$

$$\vec{\omega} = \omega_3 \hat{e}^3 + \omega_\perp \hat{e}^\perp$$

---

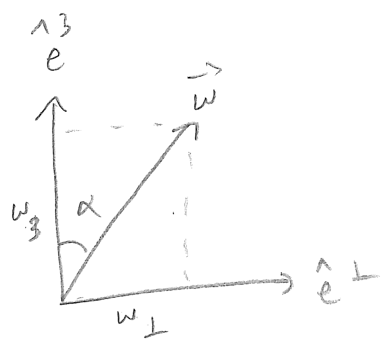
Since $\hat{e}^1$ and $\hat{e}^2$ are principle axes w/ equal moments, so is $\hat{e}^\perp$

$$\overleftrightarrow{I} \cdot \hat{e}^\perp = I_\perp \hat{e}^\perp$$

[degenerate eigenvalues
$\Rightarrow$ linear comb. of eigenvects]

---

[Now can draw in a plane]

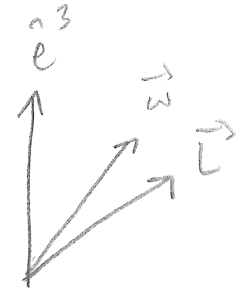



$$\omega = \sqrt{\omega_3^2 + \omega_\perp^2}$$

$$\tan \alpha = \frac{\omega_\perp}{\omega_3}$$

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

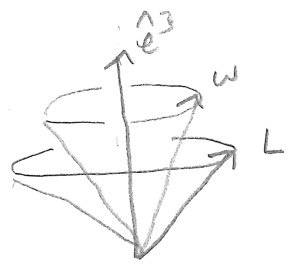
$$= I_3 \omega_3 \hat{e}^3 + I_\perp \omega_\perp \hat{e}^\perp$$




$\Rightarrow \vec{L}$ is coplanar w/ $\hat{e}^3$ and $\vec{\omega}$

---

Both $\vec{\omega}$ and $\vec{L}$ rotate around $\hat{e}^3$ w/ ang. frequency $\Omega$

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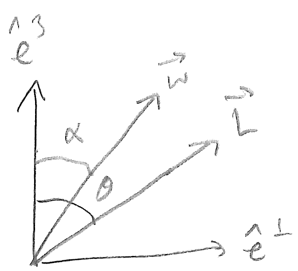



$$L_3 = I_3 \omega_3$$

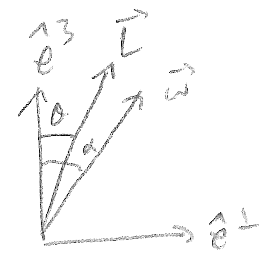
$$L_\perp = I_\perp \omega_\perp$$

$$\Rightarrow \tan\theta = \frac{L_\perp}{L_3} = \frac{I_\perp}{I_3}\frac{\omega_\perp}{\omega_3} = \frac{I_\perp}{I_3}\tan\alpha$$

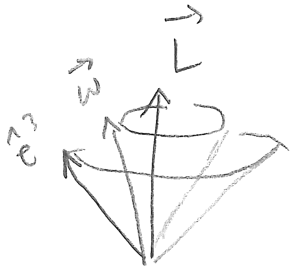
$$\boxed{I_3 \tan\theta = I_\perp \tan\alpha}$$

If $I_3 < I_\perp$ (prolate ellipsoid) $\Rightarrow \theta > \alpha$ (shown above)

If $I_3 > I_\perp$ (oblate ellipsoid) $\Rightarrow \theta < \alpha$



In space-fixed (inertial reference) frame:
in which external torque vanishes, $\vec{L}$ is constant!




$\vec{\omega}$ + $\hat{e}^3$ rotate around $\vec{L}$
(but not at frequency $\Omega$)

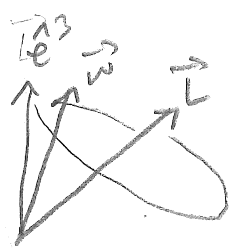
our next goal is to determine the freq.

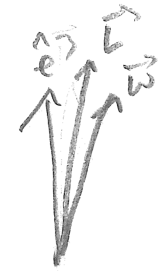
[e.g. if $I_3 = I_\perp$, $\Omega = 0$ but $\hat{e}^3$ is spinning around $\vec{\omega} = \vec{L}$ ]

Stability

If an isolated object (w/ $I_1 = I_2 = I_\perp$) is spinning w/ $\vec{\omega} \parallel \hat{e}^3$
then $\vec{L} \parallel \hat{e}^3$. Then $\vec{L}$ = const $\Rightarrow$ $\hat{e}^3$ = const
and object remains stable.

If $\vec{\omega}$ is not quite parallel to $\hat{e}^3$ then
body axis will rotate around $\vec{L}$




$I_3 < I_\perp$ (thin rod)
$\Rightarrow$ $\vec{L}$ further away from $\hat{e}^3$
(less stable)

$I_3 > I_\perp$ (disk)
$\Rightarrow$ $\vec{L}$ between $\hat{e}^3$ + $\vec{\omega}$
(more stable)