If I is a diagonal matrix, ie $\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$
the object is balanced w.r.t. $\hat{x}, \hat{y}, \hat{z}$ axes

---

If $I_{xy} \neq 0$ object is not balanced w.r.t. either $\hat{x}$ + $\hat{y}$ axes
and similarly for $I_{xz}$ + $I_{yz}$

---

However, any object, however misshapen, is always balanced wrt at least 3 axes $\hat{e}^1, \hat{e}^2, \hat{e}^3$
called principle axes
(superscripts label 3 different unit vectors)

---

The moments of inertia w.r.t. these axes, $I_1, I_2, I_3$
are called principle moments.

---

Thus if
$\vec{\omega} = \omega \hat{e}^k$ for k=1, 2, or 3, then $\vec{L} = I_k \vec{\omega}$

---

Furthermore, the principle axes are mutually perpendicular

How do we find these principle axes and moments?

In general $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$

If $\vec{\omega}$ is parallel to a principle axis $\hat{e}$ then

$$\vec{L} = I \vec{\omega}$$

Thus
$$\overleftrightarrow{I} \cdot \vec{\omega} = I \vec{\omega}$$

or
$$\overleftrightarrow{I} \cdot \hat{e} = I \hat{e}$$

Eigenvalue eqn w/ eigenvalues $I$ and eigenvector $\hat{e}$.

Write it as

$$\left[ \overleftrightarrow{I} - I \overleftrightarrow{\mathbb{1}} \right] \cdot \hat{e} = 0$$

where $\overleftrightarrow{\mathbb{1}}$ = identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

or

$$\begin{pmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{pmatrix} \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix} = 0$$

This has nontrivial solutions for $\hat{e}$ if

$$\underbrace{\det\left( \overleftrightarrow{I} - I \overleftrightarrow{\mathbb{1}} \right)}_{\text{This is a polynomial cubic in } I} = 0$$

Solve the cubic eqn for 3 values of $I$;

call them $I_1, I_2, I_3$

(In principle, some of the roots could be complex but for a symmetric matrix, they are all real.)

For each eigenvalue $I_k$, solve for the corresponding eigenvector $\hat{e}^k$.

Eigenvalue equation only determines the direction, not magnitude, of $\hat{e}^k$, so we normalize it to have length 1.

RY

Proof that eigenvectors are mutually orthogonal.

Let $\hat{e}^k$ and $\hat{e}^l$ be eigenvectors w/ distinct eigenvalues $(I_k \neq I_l)$

$$\overleftrightarrow{I} \cdot \hat{e}^k = I_k \hat{e}^k$$

$$\overleftrightarrow{I} \hat{e}^l = I_l \hat{e}^l$$

Dot w/ $\hat{e}^k$

Dot w/ $\hat{e}^l$

(1) $\hat{e}^l \cdot \overleftrightarrow{I} \cdot \hat{e}^k = I_k \hat{e}^l \cdot \hat{e}^k$

(2) $\hat{e}^k \cdot \overleftrightarrow{I} \cdot \hat{e}^l = I_l \hat{e}^k \cdot \hat{e}^l$

Claim: $\hat{e}^l \cdot \overleftrightarrow{I} \cdot \hat{e}^k = \hat{e}^k \cdot \overleftrightarrow{I} \cdot \hat{e}^l$ (3)

Proof: $\sum_{i,j} e_i^l I_{ij} e_j^k = \sum_{i,j} e_i^l I_{ji} e_j^k = \sum_{i,j} e_j^k I_{ji} e_i^l$

I is symmetric; rearrange

$= \sum_{i,j} e_i^k I_{ij} e_j^l = \hat{e}^k \cdot \overleftrightarrow{I} \cdot \hat{e}^l$ QED

relabel $i \leftrightarrow j$

Putting (1), (2), (3) together

$$I_k \hat{e}^l \cdot \hat{e}^k = I_l \hat{e}^k \cdot \hat{e}^l$$

$$\Rightarrow (I_k - I_l)\, \hat{e}^l \cdot \hat{e}^k = 0$$

But $I_k \neq I_l$ so $\hat{e}^l \cdot \hat{e}^k = 0$ QED

Proof of reality of eigenvalues

$$\overleftrightarrow{I} \cdot \hat{e}^k = I_k \hat{e}^k$$

Dot by $\hat{e}^{*k}$

$$\hat{e}^{*k} \cdot \overleftrightarrow{I} \cdot \hat{e}^k = I_k \hat{e}^{*k} \cdot e^k \qquad (1)$$

Take c.c. and use fact that $\overleftrightarrow{I}$ is real

$$\hat{e}^k \cdot \overleftrightarrow{I} \cdot \hat{e}^{*k} = I_k^* \hat{e}^k \cdot \hat{e}^{*k} \qquad (2)$$

But by previous calc: $\hat{e}^k \cdot \overleftrightarrow{I} \cdot \hat{e}^{*k} = \hat{e}^{*k} \cdot \overleftrightarrow{I} \cdot \hat{e}^k \qquad (3)$

Putting (1), (2), (3) together

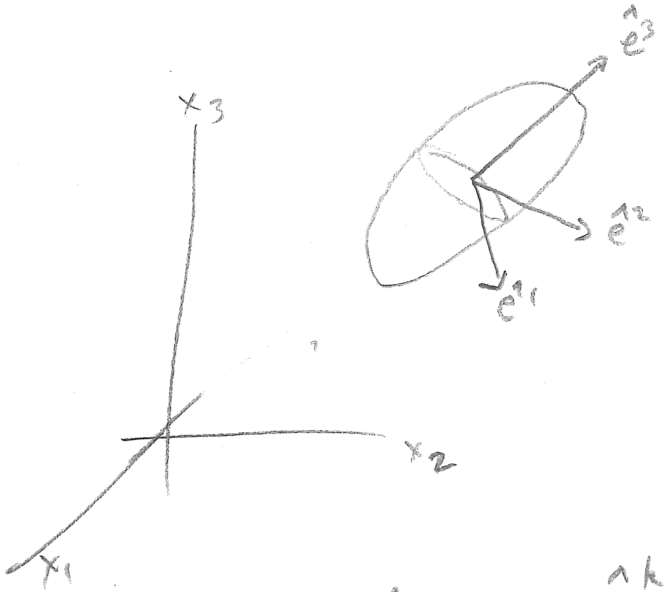
$$I_k \hat{e}^{*k} \cdot e^k = I_k^* \hat{e}^* \cdot e^k$$

$$(I_k - I_k^*) \hat{e}^{*k} \cdot e^k = 0$$

But $\hat{e}^{*k} \cdot \hat{e}^k = \sum |e_i^k|^2 > 0$

so $$I_k - I_k^* = 0$$

$$I_k^* = I_k \qquad \text{QED}$$




The principle axes $\hat{e}^k$ can be taken as the basis vectors of a coordinate system attached to the object $\Rightarrow$ "body-fixed frame"

This is not an inertial reference frame because as the object rotates, so does the coordinate system

---

We also refer to the "space-fixed frame" as the inertial reference frame w/ fixed basis vectors $\hat{x}_1 = \hat{x}$, $\hat{x}_2 = \hat{y}$, $\hat{x}_3 = \hat{z}$

An arbitrary vector $\vec{A}$ can be decomposed in either frame

$$\vec{A} = \sum A_i \hat{x}_i = A_1 \hat{x}_1 + A_2 \hat{x}_2 + A_3 \hat{x}_3$$

$$\vec{A} = \sum A_k' \hat{e}^k = A_1' \hat{e}^1 + A_2' \hat{e}^2 + A_3' \hat{e}^3$$

---

The coordinate in the two frames are linearly related

$$A_k' = \sum R_{ki} A_i$$

$$A_i = \sum (R^{-1})_{ik} A_k'$$

---

check: $$A_i = \sum_k (R^{-1})_{ik} \sum_j R_{kj} A_j$$

$$= \sum_j \underbrace{(\underbrace{R^{-1} R}_{\mathbb{1}})_{ij}}_{\delta_{ij}} A_j$$

$$= A_i$$

Given $\hat{e}^k$,

Let's find $R_{ki}$

$$A'_k = \hat{e}^k \cdot \vec{A} \qquad \text{since } \hat{e}^k \cdot \hat{e}^l = \delta_{kl}$$

$$= \hat{e}^k \cdot \left( \sum A_i \hat{x}^i \right)$$

$$= \sum_i \hat{e}^k \cdot \hat{x}^i \, A_i$$

$$= \sum_i e^k_i \, A_i$$

$e^k_i$ are components of $\hat{e}^k$ in space-fixed frame

$$\Rightarrow R_{ki} = e^k{}_i$$

row column

---

$R$ is a matrix whose rows are the components of $\hat{e}^k$

$$R = \begin{pmatrix} \hat{e}^1 \\ \hat{e}^2 \\ \hat{e}^3 \end{pmatrix} = \begin{pmatrix} e^1_1, e^1_2, e^1_3 \\ \text{etc.} \end{pmatrix}$$

Now let's find $(R^{-1})_{ik}$

$$A_i = \hat{x}_i \cdot \vec{A} \qquad \text{since } \hat{x}_i \cdot \hat{x}_j = \delta_{ij}$$

$$= \hat{x}_i \cdot \left( \sum A'_k \, \hat{e}^k \right)$$

$$= \sum_k \hat{x}_i \cdot \hat{e}^k \, A'_k$$

$$= \sum_k e^k_i \, A'_k$$

$$(R^{-1})_{ik} = e^k_i = R_{ki} = (R^T)_{ik}$$

$\Rightarrow R^{-1}$ is a matrix whose <u>columns</u> are components of $\hat{e}^k$

$$R^{-1} = R^T = (\hat{e}^1, \hat{e}^2, \hat{e}^3) = \begin{pmatrix} e^1_1 & \text{etc.} \\ e^1_2 & \\ e^1_3 & \end{pmatrix}$$

Since $R^{-1} = R^T$

$$\Rightarrow \quad R^T R = R R^T = \mathbb{1}$$

This is the definition of an <u>orthogonal</u> matrix

$3 \times 3$ orthogonal matrices form a group called $O(3)$

---

$$\det(RR^T) = \det(R) \underbrace{\det(R^T)}_{= \det R} = \det(\mathbb{1})$$

$$(\det R)^2 = 1 \qquad \Rightarrow \det R = \pm 1$$

---

If $\hat{e}^1, \hat{e}^2, \hat{e}^3$ is a right handed coordinate system, then $\det R = 1$

we call such transformations "special"

$$R \in \underset{\substack{\uparrow \\ \text{special}}}{S}\,\underset{\text{orthogonal}}{O}(3)$$

$$(RR^T)_{kl} = \sum_i R_{ki} \cdot R^T_{il}$$

$$= \sum e^k_i \cdot e^l_i$$

$$= \hat{e}^k \cdot \hat{e}^l$$

$$= \delta_{kl} \quad \checkmark$$

---

$$(R^T R)_{ij} = \sum R^T_{ik} R_{kj}$$

$$= \underbrace{\sum_k e^k_i e^k_j = \delta_{ij}}_{\text{not so obvious}}$$

Recall $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$

In space-fixed frame $L_i = \sum_j I_{ij} \omega_j$ (*)

In body-fixed frame $L'_k = \sum_l I'_{kl} \omega'_l$ (**)

---

How are $I_{ij}$ and $I'_{kl}$ related?

$L'_k = \sum_i R_{ki} L_i$ because $\vec{L}$ is a vector

$= \sum_{i,j} R_{ki} I_{ij} \omega_j$ by (*)

$= \sum_{i,j,l} R_{ki} I_{ij} (R^{-1})_{jl} \omega'_l$ because $\vec{\omega}$ is a vector

Comparing with (**) we see

$I'_{kl} = \sum_{ij} R_{ki} I_{ij} (R^{-1})_{jl}$

$= \sum_{ij} R_{ki} I_{ij} (R^T)_{jl}$

$I'_{kl} = \sum R_{ki} R_{lj} I_{ij}$

Any thing that transforms like this is called a 2nd rank tensor

---

A vector is a 1st rank tensor

$A'_k = \sum R_{ki} A_i$

---

A 3rd rank tensor $\Omega_{ijh}$ transforms as

$\Omega'_{klm} = \sum R_{ki} R_{lj} R_{mh} \Omega_{ijh}$ etc.

Compute the inertia tensor in the body-fixed frame

$$I'_{kl} = \sum R_{ki} I_{ij} (R^{-1})_{jl}$$

Rewrite a matrix eqn

$$I' = R \; I \; R^{-1}$$ ← called a "similarity transform"

$$= \begin{pmatrix} \hat{e}^1 \\ \hat{e}^2 \\ \hat{e}^3 \end{pmatrix} \cdot \overleftrightarrow{I} \cdot (\hat{e}^1, \hat{e}^2, \hat{e}^3)$$

acts on each column as an eigenvalue eqn

$$\overleftrightarrow{I} \cdot \hat{e}^k = I_k \hat{e}^k$$

$$= \begin{pmatrix} \hat{e}^1 \\ \hat{e}^2 \\ \hat{e}^3 \end{pmatrix} \cdot (I_1 \hat{e}^1, \; I_2 \hat{e}^2, \; I_3 \hat{e}^3)$$

$$= \begin{pmatrix} \hat{e}^1 \cdot I_1 \hat{e}^1 & \hat{e}^1 \cdot I_2 \hat{e}^2 & \hat{e}^1 \cdot I_3 \hat{e}^3 \\ \hat{e}^2 \cdot I_1 \hat{e}^1 & \hat{e}^2 \cdot I_2 \hat{e}^2 & \text{etc} \\ & & \end{pmatrix}$$

$$= \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

makes sense: since body-fixed coordinate axes are principle axes (balanced wrt axes)
$I'$ should be diagonal

---

We say: a similarity transformation by $R$ diagonalizes $\overleftrightarrow{I}$

Thm: any <u>symmetric</u> matrix can be diagonalized by an <u>orthogonal</u> similarity transformation

(also: hermitian, unitary)

To summarize:

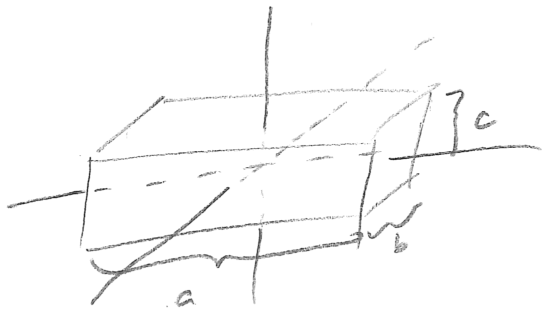
$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \qquad T = \frac{1}{2}\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$$

space-fixed $\quad L_i = \sum_j I_{ij}\,\omega_j \qquad T = \frac{1}{2}\sum_{i,j} \omega_i I_{ij}\,\omega_j$

scaler

body-fixed $\quad L'_k = I_k\,\omega_k \qquad T = \frac{1}{2}\sum_k I_k\,\omega_k^2$

If object has some obvious symmetries principle axes should be obvious

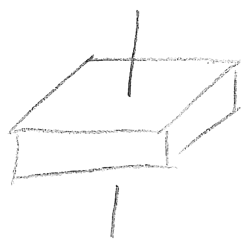



(brick)

$$I = \frac{1}{12} M \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & c^2+a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

---

If $a=b$:

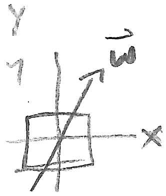


(square plate)

$$I = \frac{1}{12} \begin{bmatrix} a^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & 2c^2 \end{bmatrix}$$

$I_{xx} = I_{yy} \Rightarrow$ not only $\hat{x}$ + $\hat{y}$ but any linear comb. thereof is a principle axis

$$\overleftrightarrow{I}\cdot(\alpha\hat{x}+\beta\hat{y}) = \alpha\overleftrightarrow{I}\cdot\hat{x} + \beta\overleftrightarrow{I}\cdot\hat{y} = \alpha I_{xx}\hat{x} + \beta I_{yy}\hat{y}$$
$$= I_{xx}(\alpha\hat{x}+\beta\hat{y})$$




(any axis in xy plane is a principle axis)
(this is why proof of orthogonality fails)

Degenerate eigenvalues $\Rightarrow$ any linear comb. of corresponding eigenvectors is an eigenvector of same eigenvalue

For a cube $\overleftrightarrow{I} = \frac{1}{6} ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

↓

$I_1 = I_2 = I_3$

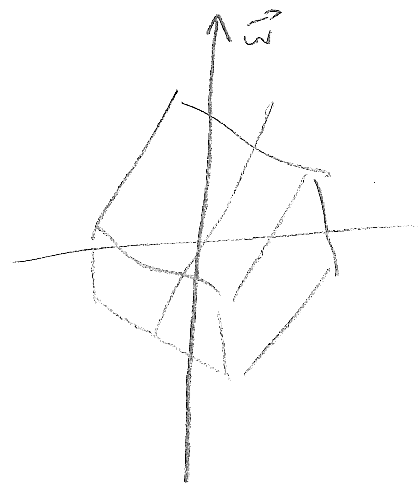
↓

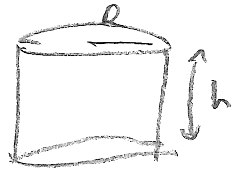
⇒ any vector is an eigenvector

~~⇒ any axis is dynamically stable~~

$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} = \frac{1}{6} ma^2 \vec{\omega}$ for any $\vec{\omega}$ through center of cube

⇒ rotation about any axis is dynamically stable



cylinder: 

For what value of $h/R$ is cylinder dynamically stable about any axis?

[ask next class]

$$\frac{1}{2} mR^2 = \frac{1}{12} mh^2 + \frac{1}{4} mR^2$$

$$\Rightarrow h = \sqrt{3} R$$