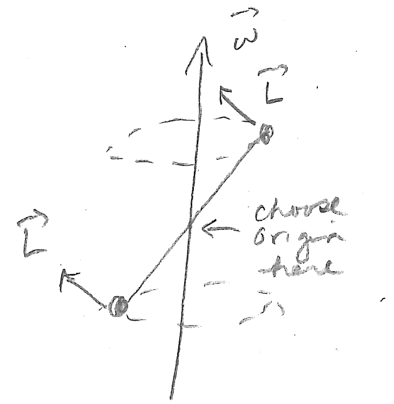
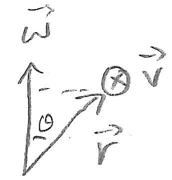
<u>Unbalanced rigid body</u> : $\vec{L}$ not parallel to $\vec{\omega}$



$$\vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

A useful formula is $\vec{v} = \vec{\omega} \times \vec{r}$



check this: direction ✓

magnitude $|\vec{v}| = \omega r \sin\theta = \omega r_\perp$ ✓

use $\hat{x}\hat{y}\hat{z}$ not $\hat{i}\hat{j}\hat{k}$ so as not to confuse indices i with $\hat{i}$

Choose z-axis for rotation $\vec{\omega} = \omega \hat{z}$

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \hat{x}(-\omega y) + \hat{y}(\omega x)$$

$$\vec{r} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ -\omega y & \omega x & 0 \end{vmatrix} = \hat{x}(-\omega x z) + \hat{y}(-\omega y z) + \hat{z}(\omega x^2 + \omega y^2)$$

$$\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i = \Big[\underbrace{\sum m_i (x_i^2 + y_i^2)}_{I_{zz}} \hat{x} + \underbrace{\sum(-m_i x_i z_i)}_{I_{xz}} \hat{y} + \underbrace{\sum(-m_i y_i z_i)}_{I_{yz}} \hat{z} \Big] \omega$$

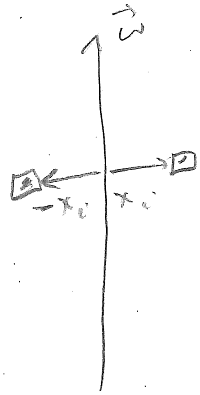
I_{zz}: moment of inertia

I_{xz}, I_{yz}: products of inertia

$\vec{L} \parallel \vec{\omega}$ iff products of inertia vanish

Q2

If object is balanced wrt z-axis



$$\sum m_i x_i z_i = 0 \Rightarrow I_{xz} = 0$$

$$\sum m_i y_i z_i = 0 \Rightarrow I_{yz} = 0$$

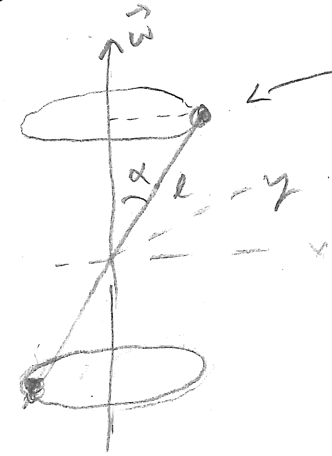
$$\Rightarrow \vec{L} = I_{zz} \hat{z} \omega = I_{zz} \vec{\omega}$$

$$I_{zz} = \sum m_i (x_i^2 + y_i^2) = \sum m_i r_{\perp i}^2$$

as before

Assume $\omega = $ const, $\vec{\omega} = \omega \hat{z}$

Baton:



$$\begin{cases} z = l \cos\alpha \\ x = l \sin\alpha \cos\omega t \\ y = l \sin\alpha \sin\omega t \end{cases}$$

$$I_{zz} = 2ml^2 \sin^2\alpha$$

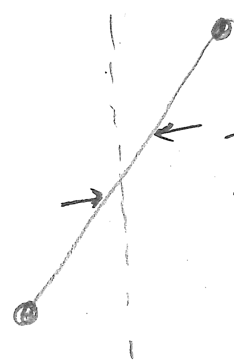
$$I_{xz} = -2ml^2 \sin\alpha \cos\alpha \cos\omega t$$

$$I_{yz} = -2ml^2 \sin\alpha \cos\alpha \sin\omega t$$

$$\vec{L} = 2ml^2\omega \sin\alpha \left[\sin\alpha\, \hat{z} - \cos\alpha(\cos\omega t\, \hat{x} + \sin\omega t\, \hat{y}) \right]$$

Not constant! torque is required to maintain rotation

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 2ml^2\omega^2 \sin\alpha \cos\alpha \left[+\sin\omega t\, \hat{x} - \cos\omega t\, \hat{y} \right]$$



These forces supply the necessary torque

[check: $\vec{r} \times \vec{F}$ is out of page]

["fighting centrifugal force"]

[washing machines!]

Rotation about an arbitrary axis $\vec{\omega} = \omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}$

$$\vec{L} = \int dm \; \vec{r} \times \vec{v}$$

$$= \int dm \; \vec{r} \times (\vec{\omega} \times \vec{r})$$

Identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

outer dot remote times adjacent
− outer dot adjacent times remote

$$\vec{L} = \int dm \left[r^2 \vec{\omega} - (\vec{r} \cdot \vec{\omega}) \vec{r} \right]$$

$$= \int dm \left[(x^2+y^2+z^2)(\omega_x \hat{x} + \omega_y \hat{y} \; \omega_z \hat{z}) - (x\omega_x + y\omega_y + z\omega_z)(x\hat{x} + y\hat{y} + z\hat{z}) \right]$$

$$= \int dm \Big\{ \left[(y^2+z^2)\omega_x - xy\omega_y - xz\omega_z \right] \hat{x} + \left[-xy\,\omega_x + (x^2+z^2)\omega_y - yz\omega_z \right] \hat{y} + \left[-xz\omega_x - yz\omega_y + (x^2+y^2)\omega_z \right] \hat{z} \Big\}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\begin{pmatrix} \int dm\,(y^2+z^2) & -\int dm\, xy & -\int dm\, xz \\ -\int dm\, xy & \int dm\,(x^2+z^2) & -\int dm\, yz \\ -\int dm\, xz & -\int dm\, yz & \int dm\,(x^2+y^2) \end{pmatrix}}_{\text{inertia matrix (or inertia tensor)}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$= \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Note that I is symmetric $I_{xy} = I_{yx}$ etc

matrix notation

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

vector notation

tensor acts on a vector to produce a vector in a (possibly) different direction

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Denote the Cartesian components of a vector $\vec{v}$ by v_i $(i = 1, 2, 3)$

$$L_i = \sum_{j=1}^{3} I_{ij}\, \omega_j$$

index notation

$$I_{ij} = I_{ji}$$

~~$\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$~~

~~tensor acts on a vector to produce a vector in a (possibly) different direction~~

~~In components $L_i = \sum_{j=1} I_{ij}\omega_j$~~

~~If we denote Cartesian components of a vector $\vec{v}$ by v_i ($i=1,2,3$)~~

Redo calculation in index notation

$$L_i = \int dm\,[\,r^2\omega_i - (\vec{r}\cdot\vec{\omega})\,r_i\,]$$

$$= \int dm\,[\,\omega_i \sum_k r_k^2 - r_i \sum_j r_j\omega_j\,]$$

$$= \sum_j \int dm\,[\,\delta_{ij}\sum r_k^2 - r_i r_j\,]\,\omega_j$$

δ_{ij} = Kronecker delta $= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

$$\sum_j \delta_{ij}\omega_j = \omega_i$$

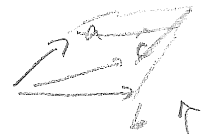
$$I_{ij} = \int dm\,[\,\delta_{ij}\sum r_k^2 - r_i r_j\,]$$

Note that $\mathbb{I}$ is a symmetric matrix: $I_{ij} = I_{ji}$

Diagonal elements $I_{ii} = \int dm\,[\,\underbrace{(\sum r_k^2) - r_i^2}_{r_{\perp i}^2}\,]$ = moments of inertia

off-diagonal elements $I_{ij} = -\int dm\, r_i r_j$ = products of inertia

Kinetic energy



volume of parallelepiped

$$T = \int \frac{1}{2}\, dm\, v^2$$

$$= \frac{1}{2}\int dm\ \vec{v}\cdot(\vec{\omega}\times\vec{r})$$

$$= \frac{1}{2}\int dm\ \vec{\omega}\cdot(\vec{r}\times\vec{v})$$

$$= \frac{1}{2}\vec{\omega}\cdot\int dm\,(\vec{r}\times\vec{v})$$

$$= \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

$$= \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

$$= \frac{1}{2}\sum_{ij}\omega_i I_{ij}\omega_j$$

Identity

$\vec{a}\cdot(\vec{b}\times\vec{c})$

$= \vec{b}\cdot(\vec{c}\times\vec{a})$

$= \vec{c}\cdot(\vec{a}\times\vec{b})$

If rotating around z-axis, reduces to

$$T = \frac{1}{2} I_{zz}\,\omega^2 = \frac{1}{2}\left(\int dm\ r_{\perp z}^2\right)\omega^2$$

Talked about Levi Civita ϵ_{ijk}

$\vec{a} = \vec{b}\times\vec{c}$

$a_i = \sum \epsilon_{ijk} b_j c_k$

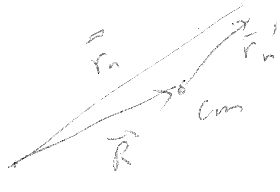
Let I_{ij} be the inertia tensor rel to ~~the~~ some origin O

Let I'_{ij} be the inertia tensor rel to cm

not necessary

$$I_{ij} = \sum m_n \left[\delta_{ij} \left(\sum_k r_{n,k}^2 \right) - r_{n,i} \cdot r_{n,j} \right]$$

$$\vec{r}_n = \vec{R} + \vec{r}_n'$$



$$I_{ij} = \sum m_n \left[\delta_{ij} \left(\sum_k (R_k + r'_{n,k})^2 \right) - (R_i + r'_{n,i})(R_j + r'_{n,j}) \right]$$

$$= \underbrace{\sum m_n \left[\delta_{ij} \left(\sum_k r'^2_{n,k} \right) - r'_{n,i} r'_{n,j} \right]}_{I'_{ij}}$$

$$+ \underbrace{\sum m_n}_{M} \left[\delta_{ij} \sum R_k^2 - R_i R_j \right]$$

$$+ \underbrace{\sum m_n \left[\delta_{ij} \sum_k 2 R_k r'_{n,k} - R_i r'_{n,j} - R_j r'_{n,i} \right]}_{0 \text{ because } \sum m_n r'_{n,l} = 0}$$

$$\boxed{I_{ij} = I'_{ij} + M \left[R^2 \delta_{ij} - R_i R_j \right]}$$

Steiner's parallel axis th

Special case

$$I_{zz} = I'_{zz} + M^2 [\underbrace{x^2 + y^2}_{r_\perp^2}]$$

$r_\perp^2 \Rightarrow$ distance btw axes