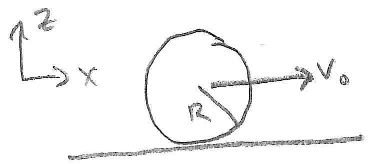
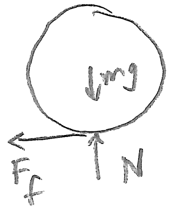
Sliding bowling ball



Hits ground w/ speed v_0 and no rotation.
When does it begin to roll?
What is its speed?
How far has it slid?



$F_f = \mu N = \mu mg$ $\qquad \mu$ = coeff of kinetic friction

Net force $\vec{F} = -\mu mg\,\hat{i}$

$\vec{F}^{ext} = m\vec{a}_{cm}$ $\qquad a_x = \frac{F_x}{m} = -\mu g$

$v_x = v_0 - \mu g t$

$x = x_0 + v_0 t - \frac{1}{2}\mu g t^2$

Torque about cm $\quad \vec{\tau} = R F_f\,\hat{j}$

$\vec{\tau} = I\vec{\alpha}$ $\qquad \alpha_y = \frac{\tau_y}{I} = \frac{R\mu mg}{I} = \frac{5}{2}\frac{\mu g}{R}$

$\omega_y = \frac{5}{2}\frac{\mu g}{R}t$

Ball starts to roll when $\quad v_x = R\omega_y \quad$ (point of contact not moving)

$v_0 - \mu g t = \frac{5}{2}\mu g t$

$t_{roll} = \frac{2}{7}\frac{v_0}{\mu g}$

Speed of ball $v_x = R\left(\frac{5}{2}\frac{\mu g}{R}\right)\left(\frac{2}{7}\frac{v_0}{\mu g}\right) = \frac{5}{7}v_0$

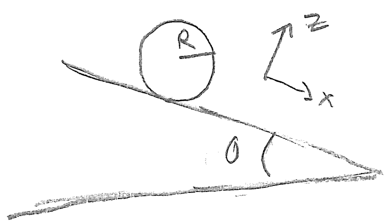
It has slid $\quad v_0 t - \frac{1}{2}\mu g t^2 = \frac{2}{7}\frac{v_0^2}{\mu g} - \frac{2}{49}\frac{v_0^2}{\mu g} = \frac{12}{49}\frac{v_0^2}{\mu g}$

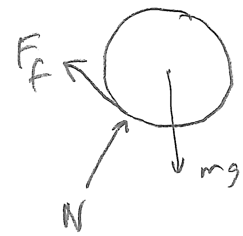
What happens after troll?

[$F_f = 0$, $\vec{v}$ + $\vec{\omega}$ are constant]

?

Cylinder rolling down incline





Net force

$$F_x = mg\sin\theta - F_f = ma_x$$

[what is F_f? $\mu mg\cos\theta$? No!] F_f is what it needs to be to keep ball rolling provided $F_f \le \mu mg\cos\theta$.

Torque about cm [accelerating]

$$\tau_y = RF_f = I\alpha_y$$

Assume rolling w/o sliding:

$$v_x = R\omega_y$$
$$a_x = R\alpha_y$$

$$\Rightarrow mg\sin\theta - F_f = mR\alpha_y = \frac{mR^2F_f}{I}$$

$$F_f = \frac{mg\sin\theta}{\left(1+\frac{mR^2}{I}\right)}$$

$$a_x = g\sin\theta - \frac{F_f}{m} = \frac{g\sin\theta}{\left(\frac{I}{mR^2}+1\right)}$$

e.g. $I = \frac{1}{2}mR^2$
$\Rightarrow a = \frac{2}{3}g\sin\theta$

Check whether cylinder will roll or slide

$$F_f \le \mu mg\cos\theta$$

$$\cancel{\sin\theta \le \left(1+\frac{mR^2}{I}\right)\mu\cos\theta}$$

$$\tan\theta \le \mu\left(1+\frac{mR^2}{I}\right)$$

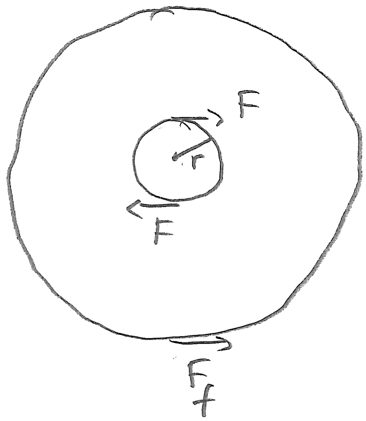
eg disk $I = \frac{1}{2}mR^2$

$\tan\theta \le 3\mu$

otherwise will start sliding and also begin to roll.

(Jan 2018)

Work done to accelerate a wheel



- $F_f = m a_{cm}$ net force
- $2rF - RF_f = I\alpha$ net torque
- $a = R\alpha$ rolling w/o slipping

$$\Downarrow$$

$$\frac{F_f}{m} = R\,\frac{(2rF - RF_f)}{I}$$

$$2mRrF = (I + mR^2)F_f$$

Frictional force does no work because point of contact moves perpendicular to force. All work is done by the torque applied to the axle

$$\Delta W = \tau\,\Delta\phi$$

$$\Delta W = 2Fr\,\Delta\phi = 2rF\omega\,\Delta t = \left(\frac{I + mR^2}{mR}\right)F_f\,\omega\,\Delta t$$

Change in kinetic energy

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(I + mR^2)\,\omega^2$$

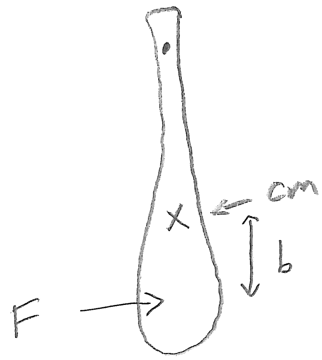
$v = \omega R$

$$\Delta K = (I + mR^2)\,\omega\,\Delta\omega$$

$$= (I + mR^2)\,\omega\,\alpha\,\Delta t \qquad \text{but } \alpha = \frac{a}{R} = \frac{F_f}{mR}$$

$$= \left(\frac{I + mR^2}{mR}\right)\omega\,F_f\,\Delta t$$, agrees w/ above

Center of percussion



A force F is applied for a short time interval Δt at a distance b below the cm. of some object.

b = impact parameter.

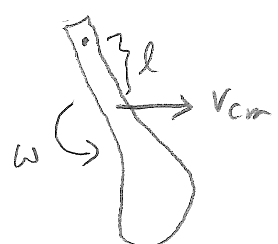
$$F = m \frac{\Delta V_{cm}}{\Delta t}$$

If object initially at rest, after the impulse $V_{cm} = \frac{F \Delta t}{m}$

The force results in a torque about cm $\tau = bF$

$$\tau = I' \frac{\Delta \omega}{\Delta t}$$

After the impulse, object rotates by $\omega = \frac{\tau \Delta t}{I'} = \frac{bF \Delta t}{I'}$



Speed of a point a distance l above cm is $V_{cm} - l\omega$

This point remains stationary if

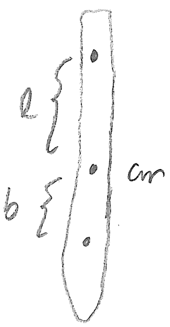
$$l\omega = V_{cm}$$

ie acts as a pivot

$$l \frac{bF\Delta t}{I'} = \frac{F \Delta t}{m}$$

$$b = \frac{I'}{ml}$$

Point of impact called center of percussion



Distance of impact point from pivot

$$b + l = \frac{(I' + ml^2)}{ml} = \frac{I}{ml}$$

where I = moment of inertia about pivot

This is same as radius of gyration, L_{eff}.

For a rod of length $L = 2l$, $I = \frac{1}{3}ML^2$

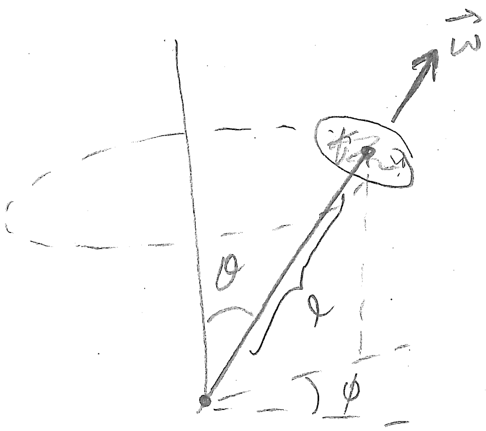
$$L_{eff} = \frac{2}{3}L$$

(For a bat, a bit longer because heavier at the end)

beyond the cm

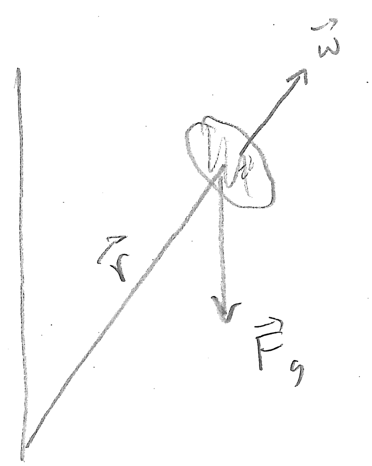
Simplified treatment of precessing top

Consider a top with its tip fixed spinning w/ angular speed $\vec{\omega}$



The external gravitational torque will cause it to precess around a vertical line w/ precession frequency $\dot{\phi}$ given approximately by $\frac{mgl}{I\omega}$

We'll derive this now in a simplified treatment and later more rigorously



$\vec{L} = I\vec{\omega}$

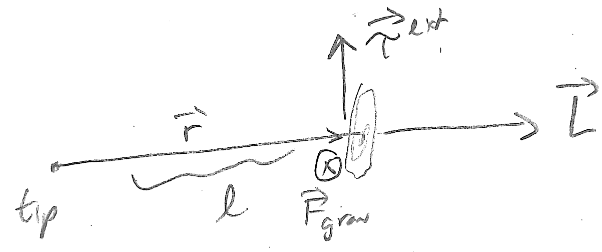
$\vec{\tau}^{\,ext} = \vec{r} \times \vec{F}_g$

($\vec{L}$ + $\vec{\tau}$ both defined wrt. fixed tip, force on tip exerts no torque.)

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{\,ext}$$

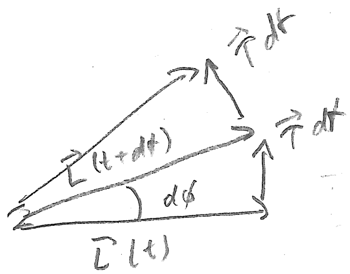
First consider $\theta = \frac{\pi}{2}$

View from overhead



$$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext} \implies \vec{L}(t+dt) = \vec{L}(t) + \vec{\tau}\, dt$$

torque causes $\vec{L}$ to change direction (& ∴ top itself)

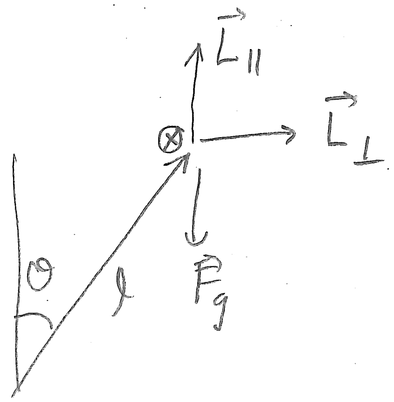


direction of top changes by $d\phi = \frac{\tau\, dt}{L}$

$$= \frac{l m g}{I\omega}\, dt$$

$$\dot{\phi} = \frac{mgl}{I\omega}$$

Now consider arbitrary θ

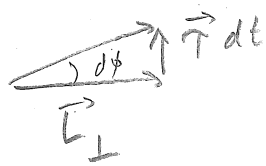


$$|\vec{\tau}| = l m g \sin\theta$$

$$L_{\perp} = I\omega \sin\theta$$

$$L_{\parallel} = I\omega \cos\theta$$

Torque causes $\vec{L}_{\perp}$ to change direction; $\vec{L}_{\parallel} = \text{const}$

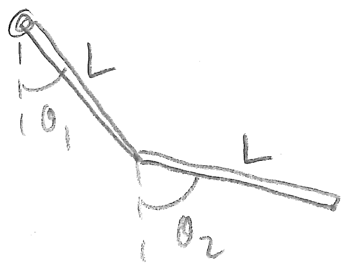


$$d\phi = \frac{mgl \sin\theta}{I\omega \sin\theta}\, dt$$

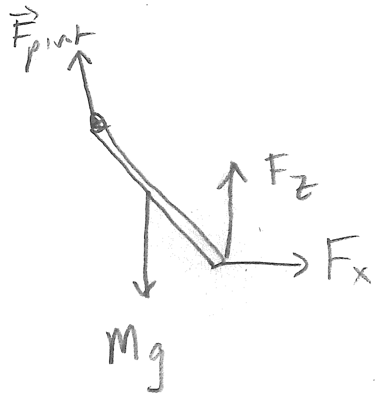
$$\dot{\phi} = \frac{mgl}{I\omega}$$

precession rate independent of angle of tilt

Double pendulum



Treat each limb as a system.
Assume each limb has length L and mass M



Let F_x, F_z be components of force exerted on upper by lower limb

Calc torque about pivot so we can ignore $\vec{F}_{pivot}$

$$\tau_y = \frac{L}{2} M g \sin\theta - L F_x \cos\theta_1 - L F_z \sin\theta_1$$

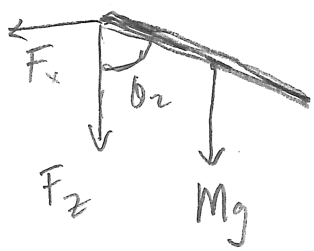
$$= \frac{dL_y}{dt} = -I\ddot{\theta}_1 \qquad \text{where } I = \frac{1}{3} M L^2$$

$$\Rightarrow \ddot{\theta}_1 = -\frac{3}{2}\frac{g}{L}\sin\theta_1 + \frac{3F_x}{ML}\cos\theta_1 + \frac{3F_z}{ML}\sin\theta_1$$

Define $\omega_0^2 = \frac{g}{L}$

$$\boxed{\ddot{\theta}_1 = -\frac{3}{2}\omega_0^2 \sin\theta_1 + 3\frac{F_x}{mL}\cos\theta_1 + 3\frac{F_z}{mL}\sin\theta_1}$$

Force on lower limb are equal & opposite



Calc torque about cm (accelerating)

→ can't calc about joint because it is not inertial.

$$\tau_y = -\frac{L}{2} F_x \cos\theta_2 - \frac{L}{2} F_z \sin\theta_2$$

$$= \frac{dL_y}{dt} = -I'\ddot{\theta}_2 \quad \text{where} \quad I' = \frac{1}{12} ML^2$$

$$\Rightarrow \boxed{\ddot{\theta}_2 = \frac{6F_x}{ML}\cos\theta_2 + \frac{6F_z}{ML}\sin\theta_2}$$

2 eqns, 4 unknowns $\theta_1, \theta_2, F_x, F_z$

use $\vec{F}^{ext} = m\vec{a}_{cm}$

let x, z be cm of lower limb

$$-F_x = M\ddot{x}$$

$$-F_z - Mg = M\ddot{z}$$

$$x = L\sin\theta_1 + \frac{L}{2}\sin\theta_2$$

$$z = -L\cos\theta_1 - \frac{L}{2}\cos\theta_2$$

Rest of calculation for problem set

- Take derivatives
- eliminate F_x + F_z from the 4 eqns
- write eqns for $\ddot{\theta}_1, \ddot{\theta}_2$