N0

Summary & review

$$\frac{d\vec{p}^{sys}}{dt} = \vec{F}^{ext} \quad \text{in IRF}$$

$$\vec{p}^{sys} = M\vec{v}_{cm}$$

$$\Rightarrow M\vec{a}_{cm} = \vec{F}^{ext}$$

If internal & external forces are conservative (or constraint)

$$E = T^{sys} + U^{int} + U^{ext} = \text{conserved}$$

can ignore for rigid body

$$T^{sys} = \frac{1}{2}Mv_{cm}^2 + T'^{sys}$$

For rigid body $T'^{sys} = \frac{1}{2}I'\omega^2$

$$\Rightarrow T^{sys} = \frac{1}{2}(Mr_{cm\perp}^2 + I')\omega^2 = \frac{1}{2}I\omega^2$$

If internal forces are central forces

$$\frac{d\vec{L}^{sys}}{dt} = \vec{\tau}^{ext} \quad \text{in IRF}$$

$$\vec{L}^{sys} = \vec{r}_{cm} \times M\vec{v}_{cm} + \vec{L}'^{sys}$$

For rigid body balanced about an axis $\vec{L}'^{sys} = I'\vec{\omega}$

$$\vec{L}^{sys} = (M r_{cm\perp}^2 + I')\vec{\omega}$$, where origin is at point on axis nearest the cm

$$= I\vec{\omega} \Rightarrow \vec{\tau}^{ext} = I\vec{\alpha}$$

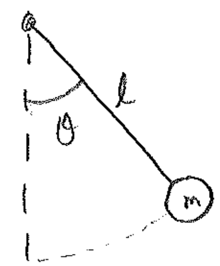
Also

$$\frac{d\vec{L}'^{sys}}{dt} = \vec{\tau}^{ext}$$ valid if origin = cm, even if cm is accelerating

$\Rightarrow \cancel{I'\vec{\alpha} = \vec{\tau}^{ext}}$

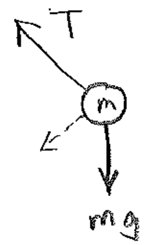
$\boxed{\text{Simple pendulum}}$: bob on a string

[done 3 ways!]



Force approach : $\vec{F} = m\vec{a}$

FBD



Tension directed along string.

We don't know its magnitude so consider components of force in tangential direction

$$F_\theta = -mg\sin\theta$$
$$ma_\theta = m\ell\ddot{\theta}$$

assume equality of grav. and inertial mass

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Small angle $\sin\theta \sim \theta$

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

ω_0 = angular frequency for small oscillations

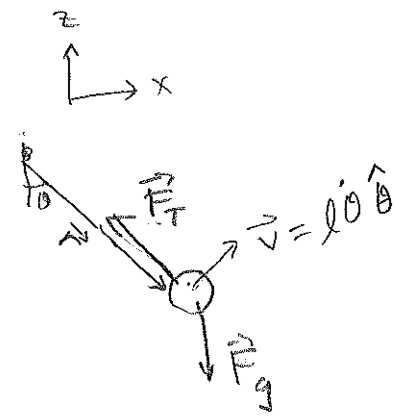
Harmonic oscillator of $\omega_0 = \sqrt{\frac{g}{\ell}}$

indep of mass!

Newton 1687 cited experiments he did to establish $m_{grav} = m_{iner}$.

Simple pendulum

Torque approach $\vec{\tau} = \frac{d\vec{L}}{dt}$



Calculate torque w/ origin at pivot.

$\vec{r} \times \vec{F}_T = 0$

$\vec{r} \times \vec{F}_{grav} = lmg \sin\theta \, \hat{y}$

Angular mom. of bob w/ orig at pivot

$\vec{L} = \vec{r} \times m\vec{v} = ml(l\dot{\theta})(-\hat{y}) = -ml^2\dot{\theta}\,\hat{y}$

$$\frac{dL_y}{dt} = \tau_y$$

$$-ml^2\ddot{\theta} = lmg\sin\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Simple pendulum

Energy approach: $T + U = \text{const}$

$\vec{v} = l\dot{\theta}\hat{\theta}$

$T = \frac{1}{2} m l^2 \dot{\theta}^2$

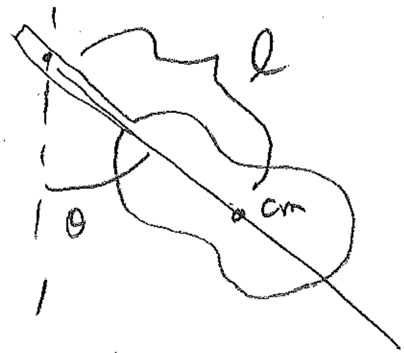
$U = -mgl\cos\theta$

$$\frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta = -mgl\cos\theta_0$$

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0)$$

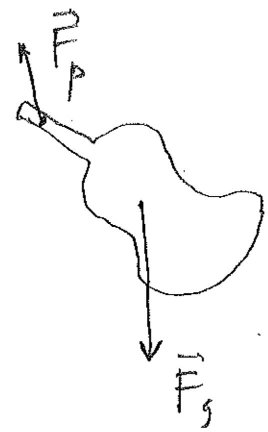
differentiate to get $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

Physical pendulum: rigid object pivoting about one point



Force approach: FBD

$\vec{F} = m\vec{a}$



[don't dwell on location of force vectors until we get to torques]

We know (a priori) neither the magnitude nor the direction of $\vec{F}_p$

($\vec{F}_p$ is not necessarily parallel to the line connecting pivot and cm)

Since F_{px} and F_{pz} unknown, force approach fails

Torque approach

choose origin at pivot.

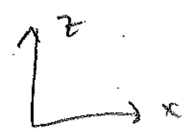
$$\vec{\tau}_p = \vec{r} \times \vec{F}_p = 0 \quad \text{because } \vec{r} = 0$$

$$\vec{\tau}_g = \int \vec{r} \times dm\,(-g\,\hat{z})$$

$$= \int (\vec{r}_{cm} + \vec{r}\,') \times dm\,(-g\,\hat{z})$$

$$= \vec{r}_{cm} \times \underbrace{\left(\int dm\right)}_{m}(-g\,\hat{z}) + \underbrace{\int \vec{r}\,'\,dm}_{0\text{, the trivial integral}} \times (-g\,\hat{z})$$

$$= -mg\,\vec{r}_{cm} \times \hat{z}$$



$$= l\,mg\sin\theta\,\hat{y}$$

Object dynamically balanced wrt. $\vec{\omega}$ axis

$\vec{L} = I\vec{\omega}$ $\quad$ I = mom of inertia wrt. pivot

$\vec{\omega} = -\dot{\theta}\,\hat{y}$

$L_y = I\omega_y = -I\dot{\theta}$

$\frac{dL_y}{dt} = -I\ddot{\theta} = \tau_y = lmg\sin\theta$

$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0 \quad \overset{\text{small } \theta}{\Rightarrow} \quad \omega_0^2 = \frac{mgl}{I} = \frac{g}{\left(\frac{I}{ml}\right)}$

$\Rightarrow$ same period as simple pendulum of length $L_{eff} = \frac{I}{ml}$

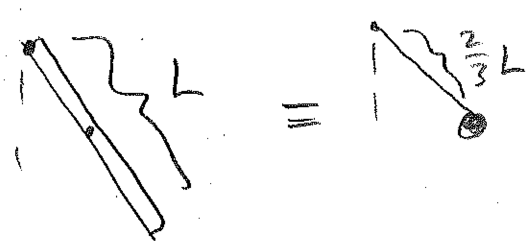
called "radius of gyration" of the physical pendulum

Uniform rod of length $L \Rightarrow I' = \frac{1}{12}mL^2$

$I = I' + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$

$l = \frac{L}{2}$

$\Rightarrow L_{eff} = \frac{2}{3}L$



Physical pendulum: energy approach

$$E = T^{sys} + U^{ext} = \text{const}$$

$$T^{sys} = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} I \dot{\theta}^2$$

$$U^{ext} = \int dm\, g h = g \int dm\, \hat{z}\cdot\vec{r}$$

$$= g\hat{z}\cdot\int dm\,(\vec{r}_{cm} + \vec{r}\,')$$

$$= g \underbrace{\hat{z}\cdot\vec{r}_{cm}}_{h_{cm}} \underbrace{\int dm}_{m} + g\hat{z} \underbrace{\int dm\, \vec{r}\,'}_{0\text{, trivial integral}}$$

$$= mgh_{cm}$$ so center of gravity = center of mass

$$= -mgl\cos\theta$$

$$E = \tfrac{1}{2} I\dot{\theta}^2 - mgl\cos\theta = -mgl\cos\theta_0$$

$$\dot{\theta}^2 = \frac{2mgl}{I}(\cos\theta - \cos\theta_0)$$

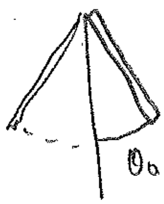
↓ differentiate

$$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0$$ as before

Period of a physical pendulum

$$E = \frac{1}{2} I \dot{\theta}^2 - mgl\cos\theta = -mgl\cos\theta_0$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \pm\sqrt{\frac{2mgl}{I}(\cos\theta - \cos\theta_0)}$$



During the upswing:

$$\int_0^{\frac{T}{4}} dt = \sqrt{\frac{I}{mgl}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}$$

Recall for small oscillations $\omega_0^2 = \sqrt{\frac{mgl}{I}}$ so

$$T = \frac{4}{\omega_0} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}$$

Small angle $\Rightarrow \cos\theta \simeq 1 - \frac{1}{2}\theta_0^2$

Phys. 3000 problem:

$$T = \frac{4}{\omega_0} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}}$$

Let $\theta = \theta_0 \sin\alpha$

$d\theta = \theta_0 \cos\alpha \, d\alpha$

$$T = \frac{4}{\omega_0} \int_0^{\frac{\pi}{2}} \frac{\theta_0 \cos\alpha \, d\alpha}{\sqrt{\theta_0^2 - \theta_0^2 \sin^2\alpha}}$$

$$= \frac{4}{\omega_0} \int_0^{\frac{\pi}{2}} d\alpha$$

$$= \frac{2\pi}{\omega_0} \quad \text{as expected}$$

Let's calculate period exactly

$$T = \frac{4}{\omega_0} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}$$

Double angle formula

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2}$$

$$T = \frac{4}{\omega_0} \int_0^{\theta_0} \frac{d\theta}{\sqrt{4\left(\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)}}$$

Let $\sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{\theta_0}{2}\right)\sin\alpha \Rightarrow \text{denom} = 2\sin\left(\frac{\theta_0}{2}\right)\cos\alpha$

$$\cos\left(\frac{\theta}{2}\right)\frac{d\theta}{2} = \sin\left(\frac{\theta_0}{2}\right)\cos\alpha \, d\alpha$$

$$d\theta = \frac{2\sin\left(\frac{\theta_0}{2}\right)\cos\alpha \, d\alpha}{\cos\left(\frac{\theta}{2}\right)}$$

$$T = \frac{4}{\omega_0} \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\cos\left(\frac{\theta}{2}\right)}$$

Same integral as before but w/ $\frac{1}{\cos\left(\frac{\theta}{2}\right)}$

$$= \frac{4}{\omega_0} \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - \left(\sin^2\frac{\theta_0}{2}\right)\sin^2\alpha}}$$

$$= \frac{4}{\omega_0} \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}}$$ where $k = \sin\frac{\theta_0}{2} \leq 1$

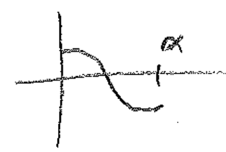
elliptic integral of the first kind. $\rightarrow \dfrac{K\left(\frac{k^2}{k^2-1}\right)}{\sqrt{1-k^2}}$

Expand in powers of k.

$$(1-k^2\sin^2\alpha)^{-\frac{1}{2}} = 1 + \frac{k^2}{2}\sin^2\alpha + \frac{3k^4}{8}\sin^4\alpha$$

$$= 1 + \frac{k^2}{2}\left(\frac{1-\cos 2\alpha}{2}\right) + \frac{3k^4}{8}\left(\frac{1-2\cos 2\alpha+\cos^2 2\alpha}{4}\right)$$

$$\text{Now} \int_0^{\frac{\pi}{2}} d\alpha \cos 2\alpha = \frac{\sin 2\alpha}{2}\Big|_0^{\frac{\pi}{2}} = 0$$



so $\cos 2\alpha$ terms drop out

$$T = \frac{4}{\omega_0}\int_0^{\frac{\pi}{2}} d\alpha \left[1 + \frac{k^2}{4} + \frac{3k^4}{8}\left(\frac{1}{4} + \frac{1}{4}\underbrace{\cos^2 2\alpha}_{\text{average to } \frac{1}{2}}\right) + \dots\right]$$

$$= \frac{2\pi}{\omega_0}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right]$$

$$= \frac{2\pi}{\omega_0}\left[1 \underbrace{+ \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2}}_{\text{corrections to small amplitude period}} + \dots\right]$$

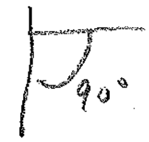
$k = \sin\left(\frac{\theta_0}{2}\right)$ N11

Let $\theta_0 = 30^\circ \Rightarrow k = 0.26$

$$T = \frac{2\pi}{\omega_0}\Big[\underbrace{1 + 0.0167 + 0.0006}_{1.0173} + \dots\Big]$$

full eval $= 1.0174$

Let $\theta_0 = 90^\circ \Rightarrow k = .0.71$



$$T = \frac{2\pi}{\omega_0}\Big[\underbrace{1 + 0.125 + 0.035}_{1.16} + \dots\Big]$$

full $1.18034 = K(\cdot)$

$K\left(\frac{1}{2}\right) = \frac{\pi}{2}(1.18034) = 1.854$

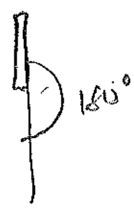
$= \frac{1.18\pi}{\omega_0}$

$= 1.18\pi\sqrt{\frac{l}{g}}$

$= \underbrace{\frac{1.18\pi}{\sqrt{2}}}_{2.621}\sqrt{\frac{d}{g}}$

later compare w/ brachisto chrone

Let $\theta_0 = 180^\circ$ $k = 1$

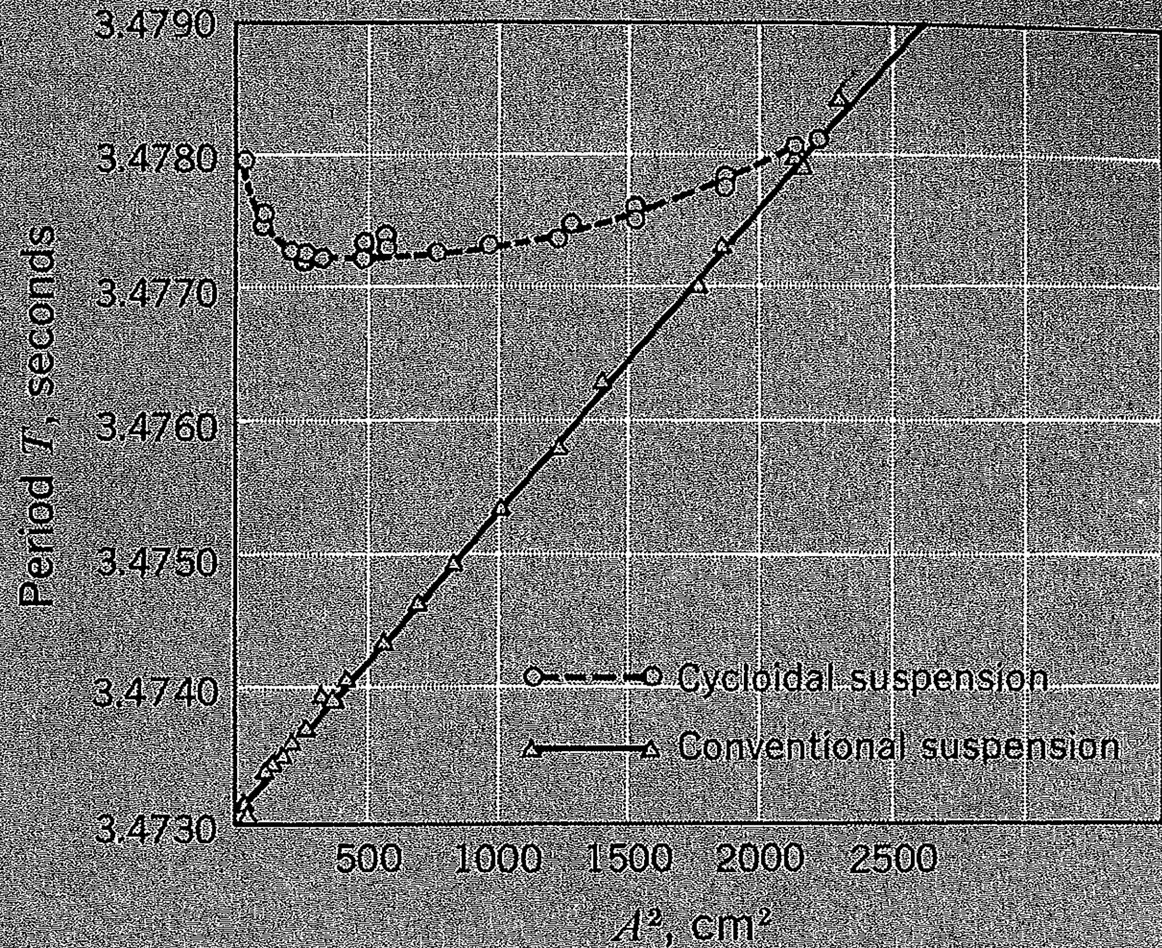


$$T = \frac{2\pi}{\omega_0}\Big[\underbrace{1 + 0.25 + 0.14}_{1.39} + \dots$$

Full $= \infty$!

$$\int \frac{d\alpha}{\sqrt{1-\sin^2\alpha}} = \int \sec\alpha \, d\alpha = \int \sec\alpha\left(\frac{\sec\alpha + \tan\alpha}{\sec\alpha + \tan\alpha}\right) d\alpha$$

$$= \ln\left|\sec\alpha + \tan\alpha\right|\Big|_0^{\frac{\pi}{2}} = \ln\left|\frac{\infty}{1}\right| = \infty.$$


Period T, seconds
3.4790
3.4780
3.4770
3.4760
3.4750
3.4740
3.4730
500
1000
1500
2000
2500
A^2, cm²
Cycloidal suspension
Conventional suspension

Replace string & bob w/ frictionless wire

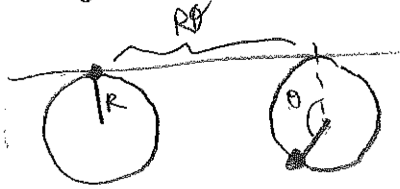
Circle $x = R\sin\theta$

$z = R(1-\cos\theta)$

Huygens: what curve gives amplitude-independent period?

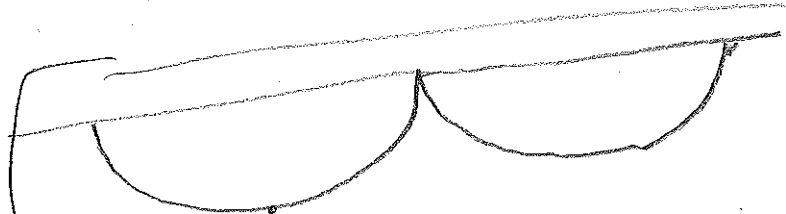
(autochrone)

answer: cycloid (point on edge of circle rolling along a line)



$x = R(\theta - \sin\theta)$

$z = -R(1-\cos\theta)$



expand around here

$\alpha = \pi + \beta$

$x = R(\beta + \sin\beta)$

$z = R(1-\cos\beta)$

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cycloidal pendulum

```
Show[ ParametricPlot[ {4 Sin[a], 4 (1 - Cos[a])}, {a, -Pi / 2, Pi / 2}],
 ParametricPlot[ {a + Sin[a], 1 - Cos[a]}, {a, -3 Pi, 3 Pi}]]
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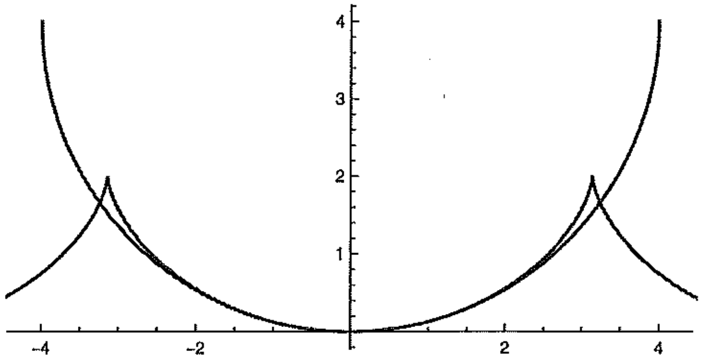
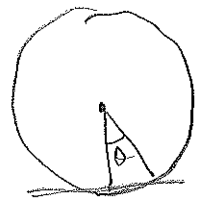


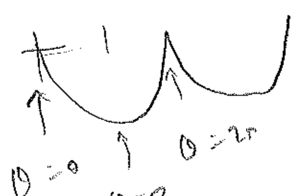
circle



$$x = R\sin\theta \approx R\theta$$
$$y = R(1-\cos\theta) \approx \tfrac{1}{2}R\theta^2 = \frac{1}{2R}x^2$$

so curvature = inverse of radius of curvature

cycloid



$$x = R(\theta - \sin\theta)$$
$$y = -R(1-\cos\theta)$$

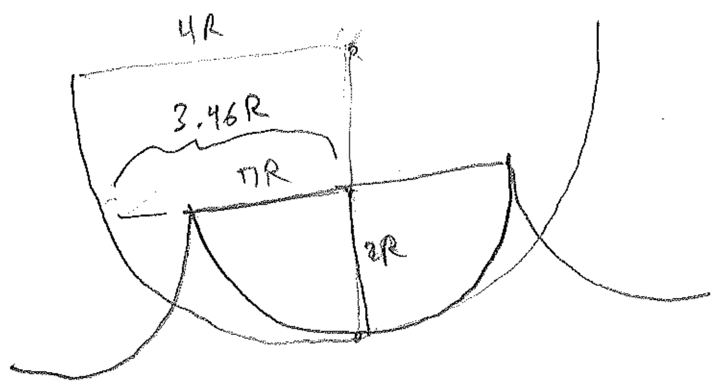
Expand around $\theta = \pi + \alpha$ $\quad \sin(\pi+\alpha) = \cos\pi\sin\alpha$
$\quad \cos(\pi+\alpha) = \cos\pi\cos\alpha$

$$x = R\pi + R(\alpha + \sin\alpha) = R\pi + 2R\alpha$$
$$y = -R(1+\cos\alpha) = -2R + \tfrac{1}{2}R\alpha^2$$

Near bottom $x' = 2R\alpha$
$$y' = \tfrac{1}{2}R\alpha^2 = \tfrac{1}{2}R\left(\frac{x'}{2R}\right)^2 = \frac{1}{8R}x'^2$$

$\Rightarrow$ radius of curvature of cycloid near bottom $= \underline{\underline{4R}}$

use this to do plot



1.732
2 3.4

$$(2R)^2 + y^2 = (4R)^2$$
$$y^2 = 12R^2 \qquad y = 2\sqrt{3}R \approx 3.46$$