Rigid body

a system of particles whose relative positions are fixed modulo an overall translation + rotation

In particular, all $|\vec{r}_{ij}|$ are const.

[eg. point masses connected by weightless rods (chemistry models of molecules)]

[atoms in a solid at their equilibrium positions]

[talk informally about d.o.f. 3N for N-pt masses but only 6 for a rigid body ⟹ euler angles]

$dE_{mech} = dW^{ext}$

if external forces are conservative $dW^{ext} = -dU^{ext}$

$$\Rightarrow d(\underbrace{E_{mech} + U^{ext}}_{E}) = 0.$$

Mechanical energy of a rigid body

$$E = \sum T_i + \sum_{i<j} U_{ij}(\vec{r}_{ij}) + U^{ext}$$

Internal constraint forces do no work & ∴ do not contribute to energy.

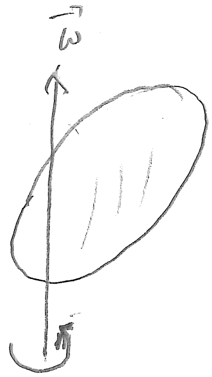
Particles at equilibrium positions $\Rightarrow \sum U_{ij} = \text{const}$ and can be ignored.

$\Rightarrow$ for a rigid body $E = T^{sys} + U^{ext}$

Rotating rigid body

Assume rigid body is rotating about some *fixed axis*

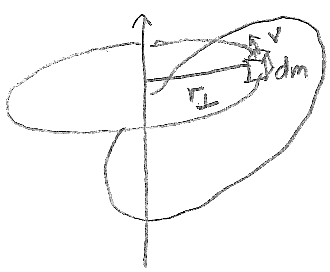
w/ angular velocity $\vec{\omega}$



$\hat{\omega}$ = axis of rotation
(direction given by r.h. rule)

$\omega = |\vec{\omega}|$ = angular speed
(need not be constant)

Break into infinitesimal elements



element dm has speed $|\vec{v}| = \omega r_\perp$

where $r_\perp$ = perpendicular distance from axis

$$T^{sys} = \int \tfrac{1}{2}(dm)v^2 = \tfrac{1}{2}\left(\int dm\, r_\perp^2\right)\omega^2$$

Define moment of inertia $I = \int dm\, r_\perp^2 = \int d^3r\, \rho r_\perp^2$

density

NB.

$$T^{sys} = \tfrac{1}{2} I \omega^2$$

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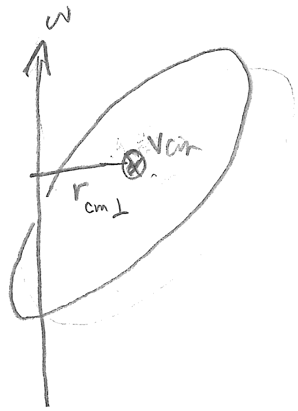
Earlier we showed $\quad T^{sys} = \frac{1}{2} M v_{cm}^2 + T'^{\,sys}$

kinetic energy in cm frame



$$|\vec{v}_{cm}| = r_{cm\perp}\,\omega$$

$$T^{sys} = \frac{1}{2} M r_{cm\perp}^2 \omega^2 + \frac{1}{2} I' \omega^2$$

mom of inertia
wrt axis through cm

$$= \frac{1}{2}\left(M r_{cm\perp}^2 + I'\right)\omega^2$$

$$\Rightarrow \quad I = M r_{cm\perp}^2 + I' \quad \text{(parallel axis theorem)}$$

[cm frame is moving & accelerating but not rotating

$\therefore$ object is rotating at same speed in cm as in rest frame]

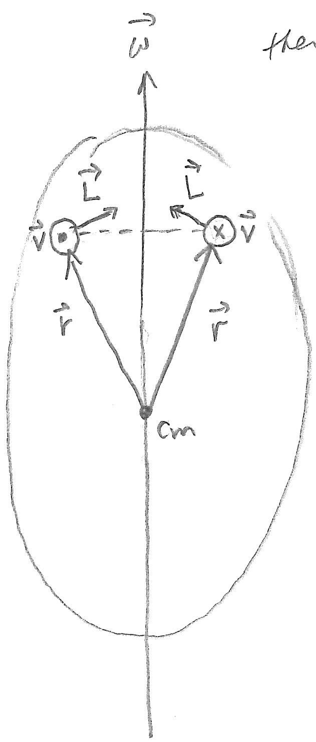
Earlier we showed $\vec{L} = \vec{r}_{cm} \times M\vec{v}_{cm} + \vec{L}'$

ang mom w.r.t. cm

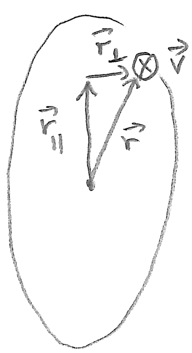
Balanced objects

Consider $\vec{L}'$ for a rotating <u>balanced</u> body

- for every element on one side of the axis there is an equal element on the opposite side



Radial components of $\vec{L}$ cancel leaving only components along $\vec{\omega}$



$$\vec{r} \times \vec{v} = (\vec{r}_\parallel + \vec{r}_\perp) \times \vec{v}$$

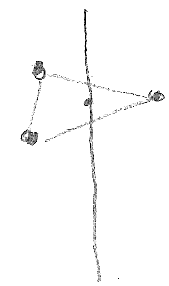
$\vec{r}_\parallel \times \vec{v}$ are radial

$\vec{r}_\perp \times \vec{v}$ are parallel to $\vec{\omega}$

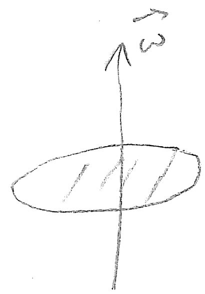
$$\Rightarrow \vec{r}_\perp \times \vec{v} = r_\perp v\, \hat{\omega}$$
$$= r_\perp (r_\perp \omega)\, \hat{\omega}$$
$$= r_\perp^2\, \vec{\omega}$$

$$\vec{L}' = \int dm\ \vec{r} \times \vec{v} = \left[\int dm\ r_\perp^2 \right] \vec{\omega} = I' \vec{\omega}$$

More generally, we call an object dynamically balanced w.r.t. axis $\hat{\omega}$ if $\vec{L}'$ is parallel to $\vec{\omega}$.



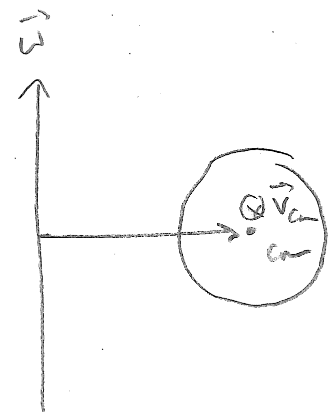
Being balanced depends not only on object but also on the axis



We will later show that for any object (however misshapen) there are at least 3 (perpendicular) axes about which it is balanced (called principle axes)

For a balanced object

$$\vec{L} = \vec{r}_{cm} \times M\vec{v}_{cm} + I'\vec{\omega}$$



Choose origin (for $\vec{L}$) to be the point on the axis closest to the cm

$$M\vec{r}_{cm} \times \vec{v}_{cm} = M r_{cm} v_{cm} \hat{\omega} = M \underbrace{r_{cm}}_{= r_{cm\perp}} (r_{cm\perp} \omega) \hat{\omega}$$

$$= M r_{cm\perp}^2 \vec{\omega}$$

$$\Rightarrow \vec{L} = (\underbrace{M r_{cm\perp}^2 + I'}_{= I}) \vec{\omega}$$

(parallel axis thm)

$\vec{L} = I\vec{\omega}$ valid for any fixed axis w/ possibly changing angular speed (provided origin is at point on axis closest to cm)

For an object balanced wrt axis $\hat{\omega}$

$$\vec{L} = I\vec{\omega}$$

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$$

$\vec{\alpha}$ = angular acceleration

Recall $\frac{d\vec{L}}{dt} = \vec{\tau}$ valid in any IRF and also in cm frame even if accelerating

$$\vec{\tau} = I\vec{\alpha}$$

analog of

$$\vec{F} = m\vec{a}$$

Perpendicular axis theorem

- applies **only** to flat objects
- assume object is in the xy plane
- assume object is balanced w.r.t. x, y and z axes.

Then $I_{zz} = I_{xx} + I_{yy}$

where I_{zz} = moment of inertia w.r.t. z-axis, etc.

[explain double index later]

[not necessarily w.r.t. cm]

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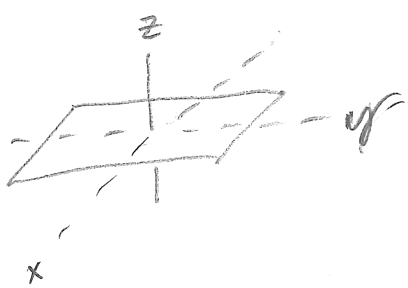
proof:



As usual $dm = \rho\, d^2r$

$$I_{zz} = \int dm\ r_\perp^2$$
$$= \int dm\ (x^2 + y^2)$$

$$I_{xx} = \int dm\ x^2$$
$$I_{yy} = \int dm\ y^2$$

QED.

Note: a solid object can often be built from a stack of flat objects

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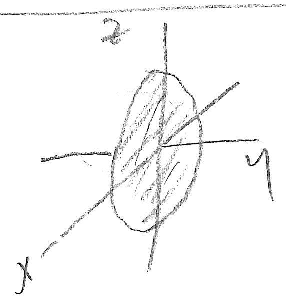
Eg.



Moment of inertia of disk w.r.t. axis in the plane of the disk I_{zz}

$$I_{zz} = I_{yy}$$
$$I_{zz} + I_{yy} = I_{xx} = \frac{1}{2} MR^2$$

$$\Rightarrow I_{zz} = \frac{1}{4} MR^2$$