Decomposition of angular momentum of a system

$$\vec{L}^{sys} = \sum m_i \vec{r}_i \times \vec{v}_i$$

Define $\vec{r}_i' = \vec{r}_i - \vec{r}_{cm}$ = position w.r.t. cm

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Observe $\sum m_i \vec{r}_i' = \underbrace{\sum m_i \vec{r}_i}_{m\vec{r}_{cm}} - \vec{r}_{cm} \underbrace{\sum m_i}_{M} = 0$

"the trivial integral"

Also $\sum m_i \vec{v}_i' = \frac{d}{dt}\left(\sum m_i \vec{r}_i'\right) = 0$

derivative of the trivial integral

$$\vec{L}^{sys} = \sum m_i (\vec{r}_{cm} + \vec{r}_i') \times (\vec{v}_{cm} + \vec{v}_i')$$

$$= \underbrace{\left(\sum m_i\right)}_{M} \vec{r}_{cm} \times \vec{v}_{cm} + \underbrace{\left(\sum m_i \vec{r}_i'\right)}_{0} \times \vec{v}_{cm} + \vec{r}_{cm} \times \underbrace{\left(\sum m_i \vec{v}_i'\right)}_{0} + \underbrace{\sum m_i \vec{r}_i' \times \vec{v}_i'}_{\vec{L}'^{\,sys}}$$

ang mom wrt origin at cm.

$$\vec{L}^{sys} = \underbrace{\vec{r}_{cm} \times M\vec{v}_{cm}}_{= \text{ ang mom of the cm}} + \underbrace{\vec{L}'^{\,sys}}_{\text{ang mom } \underline{\text{about}} \text{ the cm}}$$

$\vec{L}^{sys}$ does not depend on choice of origin iff $\vec{r}_{cm}$ is stationary

$(\Rightarrow \vec{v}_{cm} = 0)$

K2

Summary

$$\vec{P}^{sys} = M \vec{v}_{cm}$$ ← no "internal momentum"

$$T^{sys} = \frac{1}{2} M v_{cm}^2 + T'^{sys}$$

$$\vec{L}^{sys} = \vec{r}_{cm} \times M\vec{v}_{cm} + \vec{L}'^{sys}$$

change in Angular momentum of a system of particle

$$\vec{L}^{sys} = \sum \vec{L}_i$$

$$\frac{d\vec{L}^{sys}}{dt} = \sum \vec{r}_i \times \vec{F}_i$$

$$= \sum \vec{r}_i \times \left(\vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij}\right)$$

$$= \underbrace{\sum \vec{r}_i \times \vec{F}_i^{ext}}_{\substack{\text{define this to be} \\ \text{the external torque} \\ \vec{\tau}^{ext}}} + \underbrace{\sum_i \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij}}_{\sum_{pairs} (\vec{r}_i \times F_{ij} + \vec{r}_j \times \vec{F}_{ji})}$$

$$\vec{F}_{ji} = -\vec{F}_{ij}$$

$$\underbrace{(\vec{r}_i - \vec{r}_j)}_{\substack{\text{zero, for central internal force} \\ \text{because } \vec{F}_{ij} \parallel \vec{r}_{ij}}} \times \vec{F}_{ij}$$

$$\boxed{\frac{d\vec{L}^{sys}}{dt} = \vec{\tau}^{ext} \equiv \sum \vec{r}_i \times \vec{F}_i^{ext}}$$

valid of any origin in an <u>IRF</u>

Corollary: If external torque vanishes (e.g. an isolated system) $\vec{L}^{sys}$ is conserved

Corollary: if certain components of $\vec{\tau}^{ext}$ vanish, those same components of $\vec{L}^{sys}$ are conserved

Recall $\frac{d\vec{L}^{sys}}{dt} = \vec{\tau}^{ext} = \sum \vec{r}_i \times \vec{F}_i^{ext}$

$$= \sum (\vec{r}_{cm} + \vec{r}_i{}') \times \vec{F}_i^{ext}$$

$$= \vec{r}_{cm} \times \underbrace{\sum \vec{F}_i^{ext}}_{\vec{F}^{ext}} + \underbrace{\sum \vec{r}_i{}' \times \vec{F}_i^{ext}}_{\vec{\tau}'^{ext}}$$

$\vec{\tau}'^{ext}$ = ext. torque measured wrt. origin of cm

Since $\vec{L}^{sys} = M\vec{r}_{cm} \times \dot{\vec{r}}_{cm} + \vec{L}'^{sys}$

$$\frac{d\vec{L}^{sys}}{dt} = \underbrace{M\vec{r}_{cm} \times \ddot{\vec{r}}_{cm}}_{\vec{r}_{cm} \times \vec{F}^{ext}} + \frac{d\vec{L}'^{sys}}{dt}$$

Comparing previous eqn:

$$\boxed{\frac{d\vec{L}'^{sys}}{dt} = \vec{\tau}'^{ext}}$$

change in angular mom. wrt. cm equals external torque wrt. cm

→ valid even if cm frame is not inertial!

[cf Fetter & Walecka]

Summary

$$\frac{d\vec{p}^{\,sys}}{dt} = \vec{F}^{ext} \quad \text{(valid in IRF)}$$

(ditto)

$$\frac{dE^{sys}_{mech}}{dt} = \frac{dW^{ext}}{dt} = P^{ext}$$ ← power delivered by external force

$$\frac{d\vec{L}^{M'}}{dt} = \vec{\tau}^{ext}$$ ← torque caused by external force

(valid in IRF $\underline{\underline{\wedge}}$ in cm frame)