Kinetic energy of a system of particles

$$T^{sys} = \sum_{i=1}^{N} T_i$$

Decomposition of kinetic energy of a system

Recall $\vec{V}_{cm} = \frac{1}{M} \sum m_i \vec{v}_i$

Let $\vec{v}_i' = \vec{v}_i - \vec{V}_{cm}$ velocity of each particle wrt cm velocity
= velocity measured in cm frame
[possibly noninertial]

Observe that $\sum m_i \vec{v}_i' = 0$ (total mom vanishes in cm frame)

$$T^{sys} = \frac{1}{2} \sum m_i (\vec{V}_{cm} + \vec{v}_i')^2$$

$$= \frac{1}{2} \underbrace{\left(\sum m_i\right)}_{M} V_{cm}^2 + \underbrace{\left(\sum m_i \vec{v}_i' \right)}_{0} \cdot \vec{V}_{cm} + \frac{1}{2} \sum m_i v_i'^2$$

$$T^{sys} = \underbrace{\frac{1}{2} M V_{cm}^2}_{\text{kinetic energy of a particle w/ mass } M \text{ + speed } V_{cm}} + \underbrace{T'^{\,sys}}_{\text{kinetic energy of system measured in cm frame}}$$

("internal kinetic energy")

↳ thermal energy

Recall for a 2 body system

$$T^{sys} = \frac{1}{2} M v_{cm}^2 + \underbrace{\frac{1}{2} \mu v^2}$$

KE of equivalent 1-body problem
= KE of 2-body system in cm frame

Mechanical energy of a system of particles

$$dT_i = \vec{F}_i \cdot d\vec{r}_i = [\vec{F}_i^{ext} + \sum_{j\neq i} F_{ij}(\vec{r}_{ij})] \cdot d\vec{r}_i$$

$$dT^{sys} = \sum dT_i = \underbrace{\sum \vec{F}_i^{ext} \cdot d\vec{r}_i}_{dW^{ext}} + \underbrace{\sum_i \sum_{j\neq i} \vec{F}_{ij}(\vec{r}_{ij}) \cdot d\vec{r}_i}_{\sum_{\text{pairs } i<j} (\vec{F}_{ij}\cdot d\vec{r}_i + \underbrace{\vec{F}_{ji}}_{-\vec{F}_{ij}}\cdot d\vec{r}_j)}$$

work done by external forces on the system

$$\underbrace{\vec{F}_{ij} \cdot (\underbrace{d\vec{r}_i - d\vec{r}_j}_{d\vec{r}_{ij}})}_{-dU_{ij}}$$ if conservative internal force

$$\sum_i dT_i + \sum_{\text{pairs } i<j} dU_{ij} = dW^{ext}$$

Define mechanical energy $E^{sys}_{mech} = \sum_i T_i + \sum_{i<j} U_{ij}$

$$dE^{sys}_{mech} = dW^{ext}$$ (work energy thm for system)

(all internal work done is encoded in internal potential energy

Corollary: E^{sys}_{mech} conserved for an isolated system

$$E^{sys}_{mech} = T^{sys} + U^{sys}$$

$$E_{mech} = T_{cm} + T'^{sys} + U^{sys}$$

Lets include rest mass of constituents $m_i c^2$

$$E = T_{cm} + T'^{sys} + U + \sum m_i c^2$$

But for a bound state (eg proton = 3 quarks)

$$E = T_{cm} + Mc^2$$

Then

$$M = \sum m_i + \frac{T'^{sys} + U^{sys}}{c^2}$$

proton

1 GeV

< 10 MeV

U^{sys} = work done to assemble the system

- Assume internal forces fall off w/ distance sufficiently quickly that $U_{ij}(\infty) = 0$

- Start w/ m_1 at $\vec{r}_1$ and all others at ∞ (& infinitely separated)

[∞ is a big place]

- Bring in m_2, holding m_1 fixed.

Think of $\vec{F}_{21}$ as repulsive so need to exert opposite force to bring it in w/ constant velocity.

Work done on 2. is

$$W_2 = \int_{\infty}^{\vec{r}_2} (-\vec{F}_{21}) \cdot d\vec{r}_2 = \int_0 dU_{21} = U_{21}$$

→ [do this explicitly explaining all signs]

(no work done on 1 because it is held stationary)

$$\left[-\int F_x dx = -\int \frac{Kq_1q_2}{x^2} dx = \frac{Kq_1q_2}{x}\Big|_{\infty}^{x} = \frac{Kq_1q_2}{r_{12}} \right.$$

Now bring in m_3, holding m_1 & m_2 fixed

$$W_3 = \int_{\infty}^{\vec{r}_3} (-\vec{F}_{31} - \vec{F}_{32}) \cdot d\vec{r}_3 = \int dU_{31} + \int dU_{32} = U_{31} + U_{32}$$

Continue. Total work done is

$$W = U_{21} + (U_{31} + U_{32}) + (U_{41} + U_{42} + U_{43}) + \dots$$

$$= \sum_{i<j} U_{ij} = U^{sys}$$

If force is attractive, work done is negative $\Rightarrow U^{sys} < 0$.

Potential energy a sphere of radius R
+ uniformly distributed charge Q $\Rightarrow \rho = \frac{Q}{\frac{4}{3}\pi R^3}$

$$U = \sum_{i<j} U_{ij} = \frac{1}{2}\sum_{i\neq j} \frac{K q_i q_j}{r_{ij}} \longrightarrow \frac{1}{2}\int \frac{K\,\rho\, d^3r_i\, \rho\, d^3r_j}{|\vec{r}_i - \vec{r}_j|}$$

Easier to calculate by assembling it.

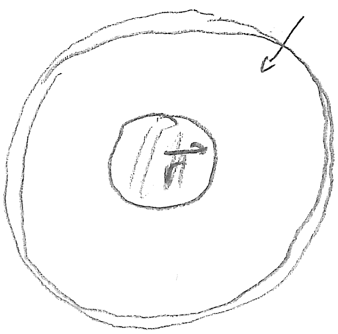
Assume have already assembled a sphere of radius r,

having done work $U(r)$.

• Assembled charge is $q(r) = Q\frac{r^3}{R^3}$

Bring in a shell of thickness dr and charge

$$dq = \rho 4\pi r^2 dr = \frac{3Q}{R^3} r^2 dr$$



work done is $\frac{K q(r)\, dq}{r} = \frac{3KQ^2 r^4 dr}{R^6}$

$$U(r+dr) = U(r) + \frac{3KQ^2}{R^6} r^4 dr$$

$$\frac{dU}{dr} = \frac{3KQ^2}{R^6} r^4$$

$$U = \frac{3}{5} KQ^2 \frac{r^5}{R^6}\Big|_0^R = \frac{3}{5}\frac{KQ^2}{R}$$

Gravitational $\Rightarrow U = -\frac{3}{5}\frac{GM^2}{R}$

energy of sun

$$\left(\frac{\cancel{1330}\ \cancel{1370}\ 1360\,W}{m^2}\right) 4\pi (1.5\times10^{11} m)^2 = \cancel{4.5}\ 3.8 \times 10^{26}\, W$$

$M_\odot = 2\times10^{30}\,kg$

$$\frac{3}{5}\left(\frac{GM^2}{R}\right) \qquad \frac{3}{5}\frac{(6.67E-11)(2E30)^2}{(7\times10^8 m)}$$

$= 2.3\times10^{41}\,J$

$\Rightarrow 6\times10^{14}$ sec

~ 20 million years

Helmholtz

white dwarfs?

neutron stars?