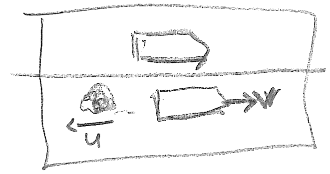
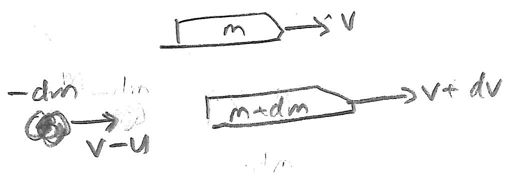
Rockets

In free space $\vec{F}^{ext} = 0 \Rightarrow \vec{p}^{sys}$ conserved

How does a rocket accelerate?



By expelling mass



Change in rocket mass $dm < 0$

mass of exhaust $-dm > 0$

u = const speed which exhaust expelled from rocket

e.g. chemical process (LOX/Kerosene) $u \sim 2500 \ m/s$ [for chemical rockets Moore says $u_{max} \sim 4500 \ m/s$]

Saturn V

$$dp^{sys} = (m+dm)(v+dv) + (-dm)(v-u) - mv$$

$$= \underbrace{mv} + m\,dv + \underline{v\,dm} + \cancel{dm\,dv}^{\text{neglect}} - \underline{v\,dm} + u\,dm - \underbrace{mv}$$

$$dp^{sys} = m\,dv + u\,dm = 0$$

$$m\frac{dv}{dt} = \underbrace{u\left(-\frac{dm}{dt}\right)}_{\text{define the quantity as thrust}} > 0$$

$$ma = F_{thrust}$$

Even though system is isolated, expulsion of exhaust acts as an effective (thrust) force on the (remaining) mass

$$m\, dv = -u\, dm$$

$$\int dv = -u \int \frac{dm}{m}$$

$$v(t) - v_0 = -u\left[\ln m(t) - \ln m_0\right]$$

$$v(t) = v_0 + u \ln\left[\frac{m_0}{m(t)}\right]$$

Change in speed depends only on exhaust speed u
and ratio of initial to final mass (not on $\frac{dm}{dt}$)

Most of initial mass is fuel

$$m_0 = M_{rocket} + m_{fuel}$$

structural constraints $\Rightarrow \frac{m_{fuel}}{m_{rocket}} \lesssim 10-20$, $\ln(20) \sim 3$

$u = 2500\,m/s$ $\Rightarrow v_{max} \lesssim 7500\,m/s$ [disregarding gravity]

but $v_{escape} \sim 11{,}000\,m/s$

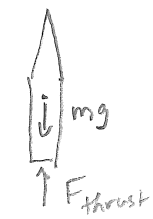
need multiple stages

Ascent under gravity

$$\frac{dp^{sys}}{dt} = F^{ext}$$

$$m\frac{dv}{dt} + u\frac{dm}{dt} = -mg$$

$$m\frac{dv}{dt} = u\left(-\frac{dm}{dt}\right) - mg$$



Assume mass is expelled at a constant rate

$$\frac{dm}{dt} = -R = \text{const} \quad \rightarrow F_{thrust} = uR$$

$$m\frac{dv}{dt} = uR - mg$$

Require $uR > mg$ for lift off

$$\frac{dv}{dt} = \frac{uR}{m} - g$$

$$m = m_0 - Rt$$

$$\frac{dv}{dt} = \frac{uR}{(m_0 - Rt)} - g$$

acceleration increases as rocket loses mass.

[Problem] find v and z by integrating

$$v = v_0 - gt + u\log\left(\frac{m_0}{m_0 - Rt}\right)$$

$$z = z_0 + v_0 t - \frac{1}{2}gt^2 + ut - \frac{u}{R}(m_0 - Rt)\log\left(\frac{m_0}{m_0 - Rt}\right)$$

Saturn V rocket used for Apollo missions

- Initial mass $m_0 = 2.8 \times 10^6$ kg [3100 tons] [French, p. 327]
- mass of first stage fuel $= 2.1 \times 10^6$ kg [$\frac{3}{4}$ of total]
- Initial weight $m_0 g = 2.74 \times 10^7$ N $= 6.2$ million lbs [1 lb = 4.45 N]

exhaust speed $u = 2500 \frac{m}{s}$

For liftoff, need $R > \frac{m_0 g}{u} = 11{,}000 \frac{kg}{s}$

Saturn V had $R = 14{,}000 \frac{kg}{s}$ [15 tons/sec] [French, p 327]

$\Rightarrow$ thrust $uR = 3.5 \times 10^7$ N $= 7.9$ million lbs

(just a bit more than weight)

Time of first stage burn $t = \frac{2.1 \times 10^6 \text{ kg}}{14000 \text{ kg/s}} = 150$ sec $\sim 2\frac{1}{2}$ minutes [French p 327]

after which $m(150 \text{ sec}) = \frac{1}{4} m_0$

[show video]

Prob: calc: initial acceleration (2.5 m/s^2)

acceleration near end of first stage burn (40 m/s^2)

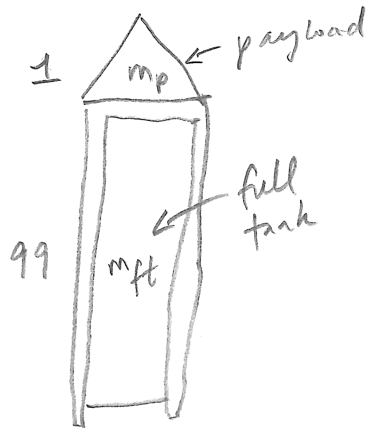
speed at end of first stage burn (2000 m/s)

height at end of first stage burn (90 km)

(assume vertical ascent)

• Max velocity for single stage rocket $\sim u \log\left(\frac{m_i}{m_f}\right)$
mass ratio is limited by structural constraints

Most of mass is fuel, but need container to hold it



Let $m_p = 1$, $m_{ft} = 99$

$$M_i = m_p + m_{ft} = 100$$

Suppose ratio of full tank to empty is 9:1

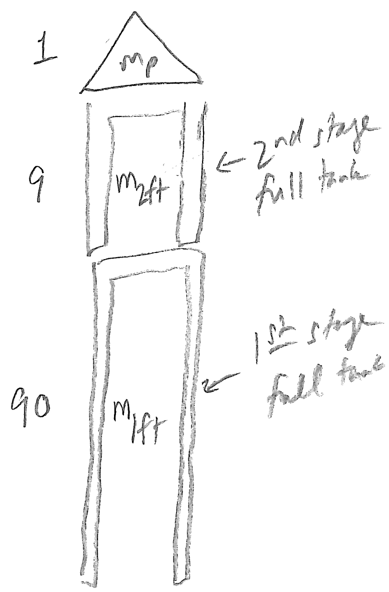
$$m_{et} = 11$$

$$M_f = m_p + m_{et} = 12$$

$$\log\left(\frac{M_i}{M_f}\right) = \log\left(\frac{100}{12}\right) \sim 2.1$$

Final velocity $v = v_0 + u \log\left(\frac{M_i}{M_f}\right) \sim v_0 + 2.1\, u$

Instead split fuel into two stages



$$m_p = 1$$

$$m_{2ft} = 9 \qquad m_{2et} = 1$$

$$m_{1ft} = 90 \qquad m_{1et} = 10$$

Initial rocket mass

$$M_i = m_p + m_{2ft} + m_{1ft} = 100$$

At end of first stage burn

$$M_{1f} = m_p + m_{2ft} + m_{1et} = 1 + 9 + 10 = 20$$

$$\log\left(\frac{M_i}{M_{1f}}\right) = \log\left(\frac{100}{20}\right) = 1.6$$

$$V_1 = V_0 + u \log\left(\frac{M_i}{M_{1f}}\right)$$

Now eject empty 1st stage tank

$$M_{2i} = m_p + m_{2ft} = 1 + 9 = 10$$

At end of 2nd stage burn

$$M_{2f} = m_p + m_{2et} = 1 + 1 = 2$$

$$\log\left(\frac{M_{2i}}{M_{2f}}\right) = \log\left(\frac{10}{2}\right) = 1.6$$

$$V_2 = V_1 + u \log\left(\frac{M_{2i}}{M_{2f}}\right)$$

$$V_2 = V_0 + u \log\left(\frac{M_{1i}}{M_{1f}}\right) + u \log\left(\frac{M_{2i}}{M_{2f}}\right)$$

$$= V_0 + 3.2\,u.$$