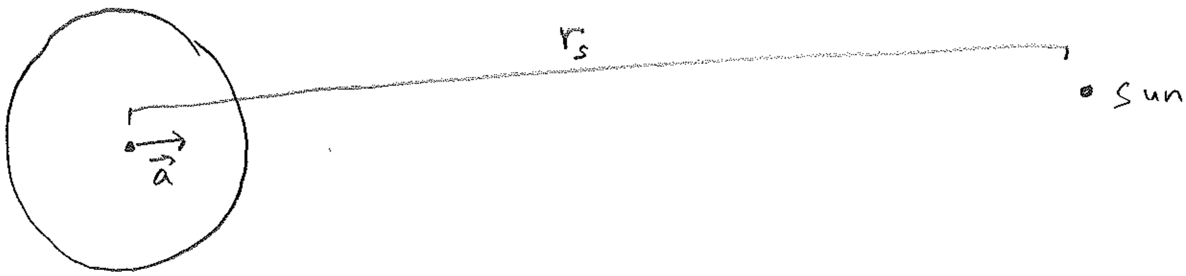
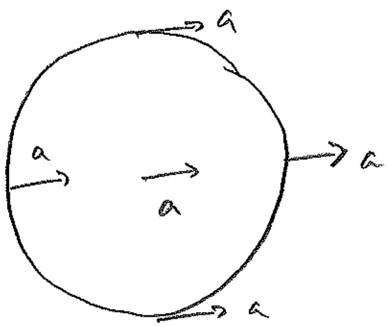
$\boxed{\text{Tides}}$ caused by sun and the moon (recognized in antiquity)

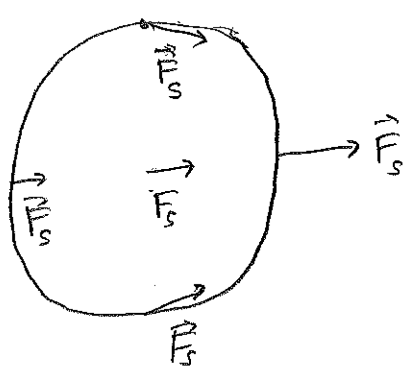
Center of mass of earth accelerates toward sun $\Rightarrow a = \dfrac{GM_s}{r_s^2}$



Treat earth as a rigid, nonrotating body:
all parts accelerate in the same direction



But force of the sun on different parts of the earth differs slightly



This sets up stresses in the earth so that

$$\vec{F}_s + \vec{F}_{other} = m\vec{a} \quad \text{for each part}$$

Accelerating reference frames

Let S be an inertial reference frame

A particle in S subject to net force $\vec{F}$ obeys

$$m\vec{a} = \vec{F}$$

The position of the particle relative to the cm is

$$\vec{r}' = \vec{r} - \vec{r}_{cm}$$
$$\vec{v}' = \vec{v} - \vec{v}_{cm}$$
$$\vec{a}' = \vec{a} - \vec{a}_{cm} = \text{acceleration of particle in the cm frame}$$

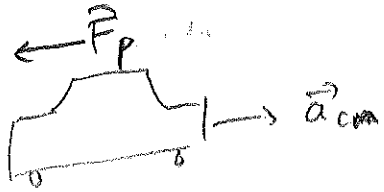
$$m\vec{a}' = m\vec{a} - m\vec{a}_{cm}$$
$$= \vec{F} - m\vec{a}_{cm}$$

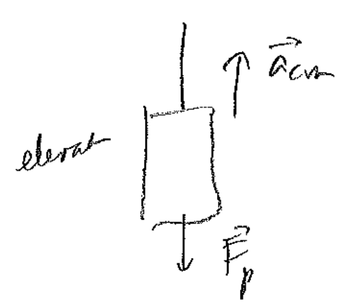
2nd law does not hold in cm frame

But if we treat $-m\vec{a}_{cm}$ as a pseudo force $\vec{F}_p$

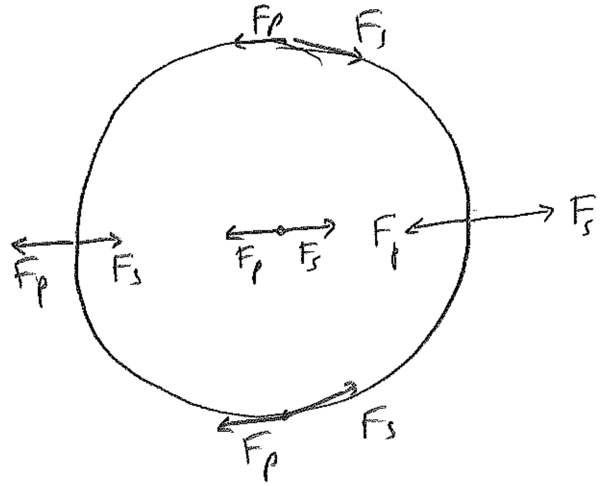
$$m\vec{a}' = \vec{F} + \vec{F}_p$$

We can pretend the 2nd law holds, where net force is a sum of real and pseudo forces

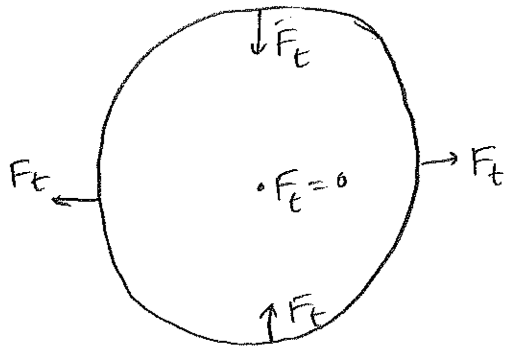




In the accelerating cm frame of the earth

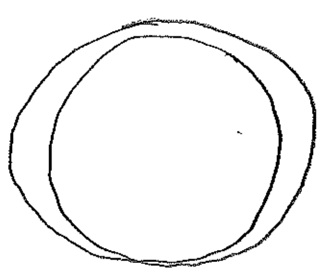


Define the tidal force $\vec{F}_t$ as the (vector) sum of sun's force and pseudoforce

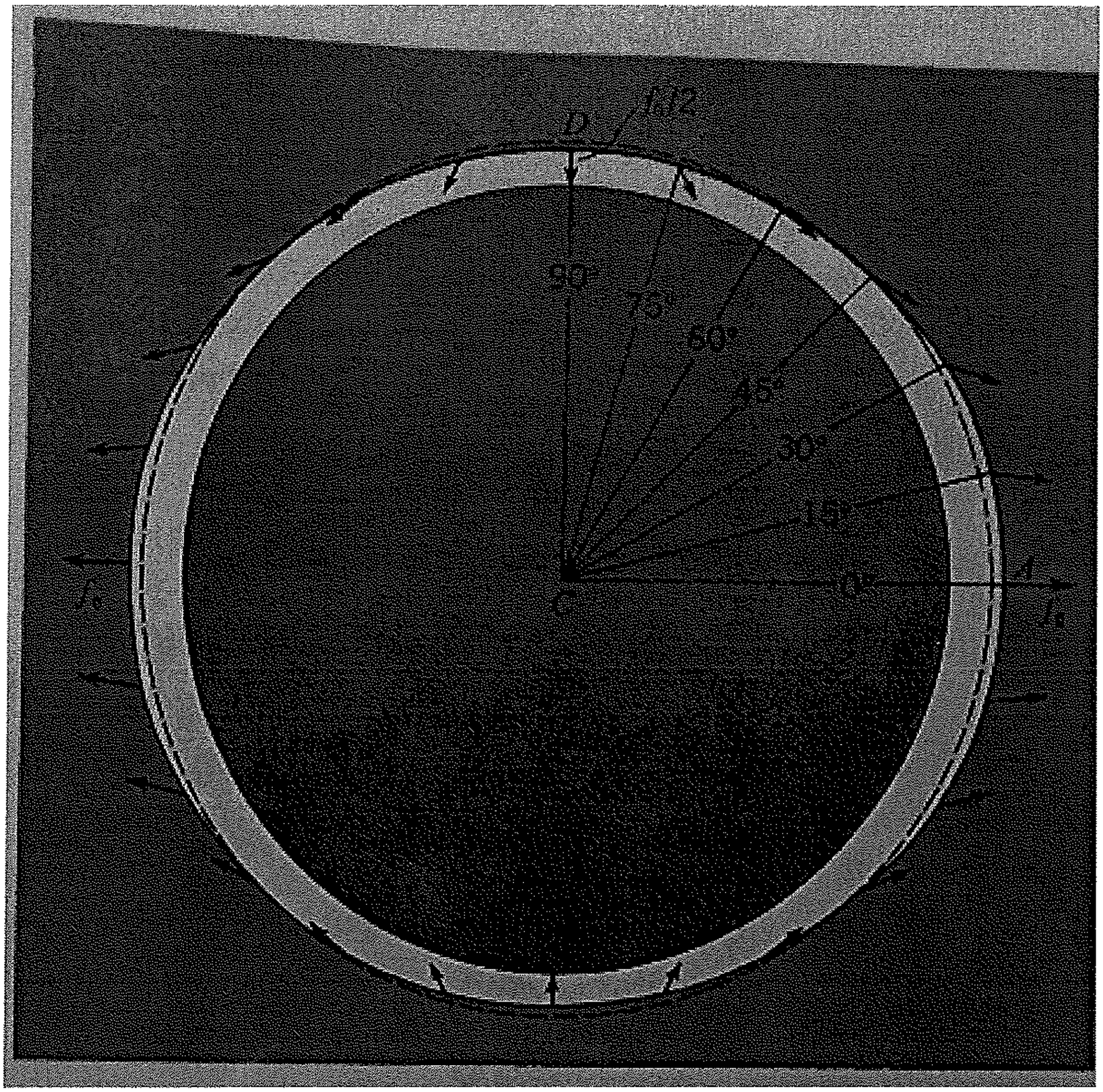


[show fig. from French]

Tidal force exerts differential stresses on different parts of the earth, causing it to deform slightly especially the water envelope



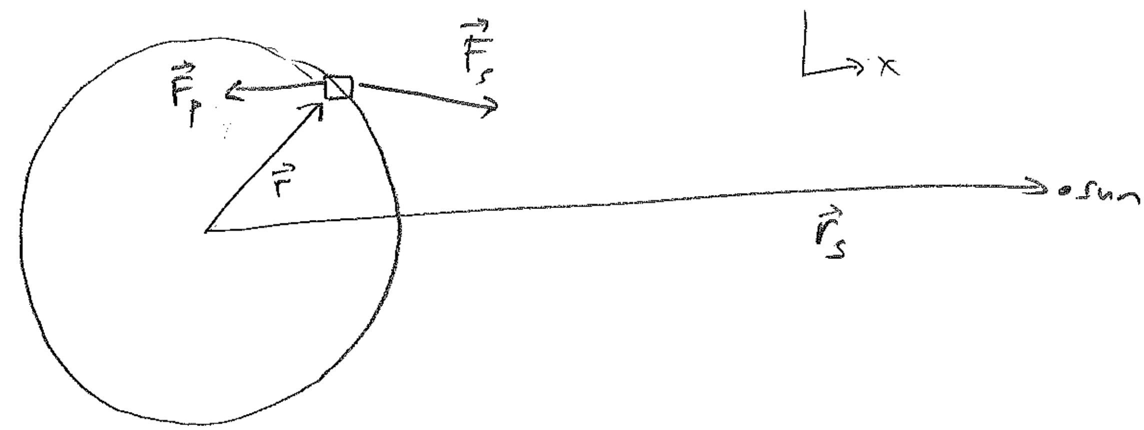
Because of earth's rotation, there are two high tides a day [first explained by Newton]



D
$f_t/2$
90°
75°
60°
45°
30°
15°
0°
A
C
f_c
f_t

[we follow (more or less) John Taylor's excellent presentation, p. 333]

Consider a point mass on the surface of the earth



$$\vec{F}_t = \vec{F}_s + \vec{F}_p$$

$$\vec{F}_s = -\frac{GM_s m(\vec{r}-\vec{r}_s)}{|\vec{r}-\vec{r}_s|^3}$$ [check sign]

$$\vec{F}_p = -\frac{GM_s m\,\hat{x}}{r_s^2}$$

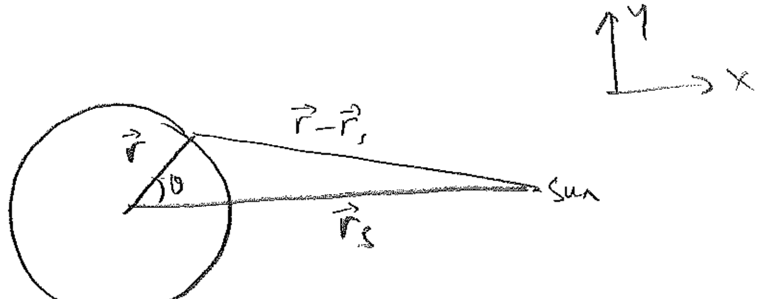
Both forces are conservative so

$$\vec{F}_s = -\vec{\nabla}U_s \Rightarrow U_s = -\frac{GM_s m}{|\vec{r}-\vec{r}_s|}$$

$$\vec{F}_p = -\vec{\nabla}U_p \Rightarrow U_p = \frac{GM_s m\,x}{r_s^2}$$

$$\vec{F}_t = -\vec{\nabla}U_t \quad \text{where } U_t = U_s + U_p$$

Q5



Now $(\vec{r}-\vec{r}_s)^2 = r^2 + r_s^2 - 2\vec{r}\cdot\vec{r}_s = r_s^2\left(1 - \frac{2r}{r_s}\cos\theta + \frac{r^2}{r_s^2}\right)$

$$U_t = -\frac{GM_s m}{r_s}\left[1 - \frac{2r}{r_s}\cos\theta + \frac{r^2}{r_s^2}\right]^{-\frac{1}{2}} + \frac{GM_s m\,(r\cos\theta)}{r_s^2}$$

$$= -\frac{GM_s m}{r_s}\left[1 + \cancel{\frac{r}{r_s}\cos\theta} - \frac{1}{2}\frac{r^2}{r_s^2} + \frac{3r^2}{2r_s^2}\cos^2\theta - \cancel{\frac{r}{r_s}\cos\theta}\right]$$

$$= -\frac{GM_s m}{r_s}\left(1 + \frac{r^2}{r_s^2}\Big[\underbrace{\frac{3}{2}\cos^2\theta - \frac{1}{2}}_{P_2(\cos\theta)}\Big]\right) \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$= -\frac{GM_s m}{r_s}\left(1 + \frac{1}{r_s^2}\Big[\underbrace{\frac{3}{2}x^2 - \frac{1}{2}(x^2+y^2+z^2)}_{x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2}\Big]\right)$$

$$F_{tx} = -\frac{\partial U_t}{\partial x} = \frac{2GM_s m}{r_s^3}\,x$$

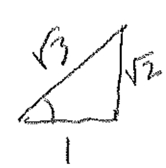
$$F_{ty} = -\frac{\partial U_t}{\partial y} = -\frac{GM_s m}{r_s^3}\,y$$

refer to picture in French

optional

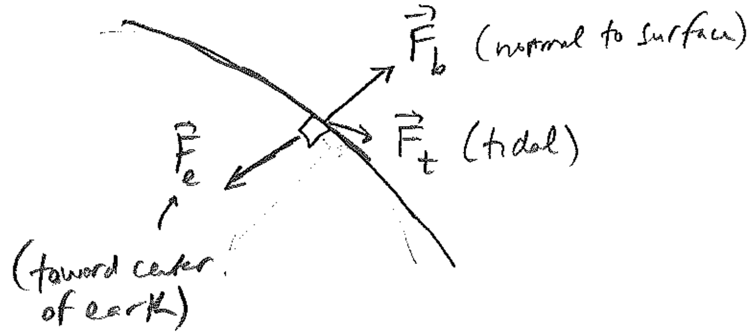
$0 = \vec{r}\cdot\vec{F} = \cos\theta\cdot 2\cos\theta + \sin\theta\,(-\sin\theta) = 3c^2 - 1 = 0 \qquad \cos\theta = \frac{1}{\sqrt{3}}$

$\theta = 54.7^\circ$



Consider a parcel of water on the surface of the ocean.

$$\vec{F} = \vec{F}_e + \vec{F}_t + \vec{F}_b$$ where $\vec{F}_b$ = buoyant force supporting the parcel is <u>normal</u> to the ocean's surface



$$\vec{F}_e = -\frac{GM_e m}{r^2}\hat{r} = -\nabla U_e \Rightarrow U_e = -\frac{GM_e m}{r}$$

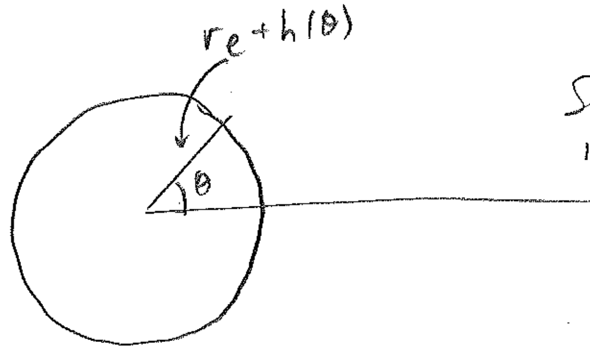
$$\Rightarrow \vec{F} = -\vec{\nabla}(U_e + U_t) + \vec{F}_b$$

In equilibrium $\vec{F} = 0$ so $\vec{\nabla}(U_e + U_t) = \vec{F}_b$

Since $\vec{\nabla}(U_e + U_t)$ is normal to surface,

the surface is an equipotential surface.

$U_e + U_t$ = const at every point on surface



Suppose the ocean surface is higher than r_e by amount $h(\theta)$

$$U_e = -\frac{GM_e m}{r_e + h(\theta)} \doteq -\frac{GM_e m}{r_e}\left[1 - \frac{h(\theta)}{r_e} + \dots\right]$$

$$= -\frac{GM_e m}{r_e} + m\underbrace{\left(\frac{GM_e}{r_e^2}\right)}_{g} h(\theta) + \dots \quad \rightarrow \text{just } mgh$$

Let $U_e + U_t = \text{const}$, i.e. independent of θ

$$-\frac{GM_e m}{r_e} + \frac{GM_e m}{r_e^2} h(\theta) - \frac{GM_s m}{r_s} - \frac{GM_s m r_e^2}{r_s^3} P_2(\cos\theta) = \text{const}$$

$$-\frac{GM_e m}{r_e} - \frac{GM_s m}{r_s} + \frac{GM_e m}{r_e^2}\left[h(\theta) - \frac{m_s}{m_e}\frac{r_e^4}{r_s^3} P_2(\cos\theta)\right] = \text{const}$$

$$\Rightarrow h(\theta) = \frac{m_s}{m_e}\frac{r_e^4}{r_s^2} P_2(\cos\theta)$$

High tide: $\theta = 0$, $P_2(1) = 1$, $\quad h(0) = \frac{m_s}{m_e}\frac{r_e^4}{r_s^2}$

Low tide: $\theta = \frac{\pi}{2}$, $P_2(0) = -\frac{1}{2}$, $\quad h(\frac{\pi}{2}) = -\frac{m_s}{2m_e}\frac{r_e^4}{r_s^2}$

Tidal variation: $\Delta h = \boxed{\frac{3}{2}\frac{m_s}{m_e}\frac{r_e^4}{r_s^3}}$

For sun: $\Delta h = 25$ cm

For moon $\Delta h = \frac{3}{2} \frac{M_m}{M_e} \frac{r_e^4}{r_m^3} = 54$ cm

When moon is full or new, effects add 79cm
"spring tide"

when moon is half, effects cancel $\Rightarrow$ 29 cm
"neap tide"