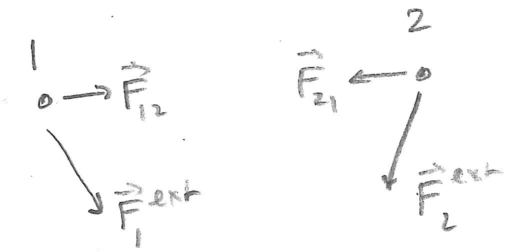
Two-body problem

A system of 2 point masses
interacting with one another
and also possibly acted on by external forces [implying at least a 3rd body]



[e.g. earth/moon/sun]

Standard convention: $\vec{F}_{12}$ = force on 1 by 2

$\vec{F}_{21}$ = force on 2 by 1

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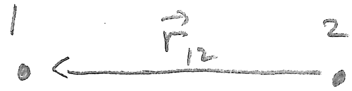
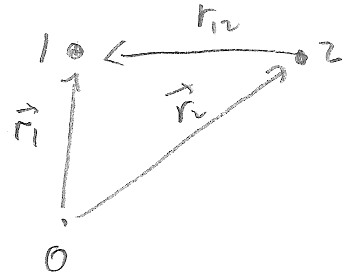
e.o.m.

$$\begin{cases} m_1 \ddot{\vec{r}}_1 = \vec{F}_{12} + \vec{F}_1^{ext} \\ m_2 \ddot{\vec{r}}_2 = \vec{F}_{21} + \vec{F}_2^{ext} \end{cases}$$



$\vec{r}_{12}$ = position of 1 relative to 2

$\vec{F}_{12}$ depends on the relative position $\vec{r}_{12}$ only
not on $\vec{r}_1$ + $\vec{r}_2$ separately (which depend on an arbitrary choice of origin)



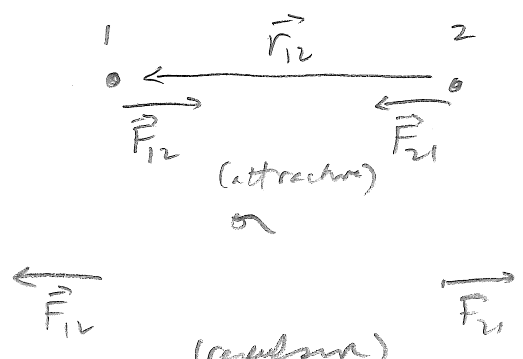
$\vec{r}_2 + \vec{r}_{12} = \vec{r}_1 \quad \Rightarrow \quad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$

Newton's 3rd law: internal forces are equal & opposite

$$\vec{F}_{21} = -\vec{F}_{12}$$

does not apply universally (e.g. not for velocity-dependent forces such as magnetism) but valid for many position-dependent forces (electrostatic, gravity, contact forces)

Central forces: forces that obey 3rd law and also $\vec{F}_{12}$ is parallel (or antiparallel) to $\vec{r}_{12}$



"strong form of the 3rd law"

[Rewrite e.o.m.]

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_{12}(\vec{r}_{12}) + \vec{F}_1^{ext}(\vec{r}_1)$$

$$m_2 \ddot{\vec{r}}_2 = -\vec{F}_{12}(\vec{r}_{12}) + \vec{F}_2^{ext}(\vec{r}_2)$$

2 coupled o.d.e's

Add the eqns

$$m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = \vec{F}_1^{ext}(\vec{r}_1) + \vec{F}_2^{ext}(\vec{r}_2)$$

Internal forces cancel by 3rd law

Let's find an eqn for $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ by subtracting the eqns

$$\ddot{\vec{r}}_{12} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12}(\vec{r}_{12}) + \left(\frac{1}{m}\vec{F}_1^{ext}(\vec{r}_1) - \frac{1}{m_2}\vec{F}_2^{ext}(\vec{r}_2)\right)$$

This suggests some definitions

Let $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ or $\mu = \frac{m_1 m_2}{m_1 + m_2}$

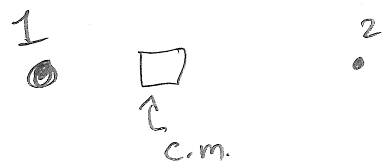
μ is called the reduced mass of the system

First eqn suggests we define

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} = \text{position of center of mass}$$

(mean position weighted by masses)

$M = m_1 + m_2$



Then $M\ddot{\vec{r}}_{cm} = m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2$

$$M\ddot{\vec{r}}_{cm} = \vec{F}_1^{\,ext}(\vec{r}_1) + \vec{F}_2^{\,ext}(\vec{r}_2)$$

This says acceleration of center of mass depends only only external forces (not internal)
[but external forces are evaluated at $\vec{r}_1$ + $\vec{r}_2$ not at $\vec{r}_{cm}$]

$$\ddot{\vec{r}}_{12} = \frac{1}{\mu}\vec{F}_{12}(\vec{r}_{12}) + \left[\frac{1}{m_1}F_1^{ext}(\vec{r}_1) - \frac{1}{m_2}F_2^{ext}(\vec{r}_2)\right]$$

[Note to instructor: if ext forces are uniform in space but not proportional to m (e.g. hydrogen atom in const $\vec{E}$ field) the eqns decouple, but r_{12} still depends on the external forces as well as internal. cf Sommerfeld Atomic Structure, p. 276]

Now let's suppose that

(1) either external forces are absent

(2) or they are due to a uniform gravitational field $\vec{g}$

ie $\vec{F}_1^{ext} = m_1\vec{g}$, $\vec{F}_2^{ext} = m_2\vec{g}$

Then term in square brackets cancels and

$$\begin{cases} M\ddot{\vec{r}}_{cm} = m_1\vec{g} + m_2\vec{g} = M\vec{g} \\ \ddot{\vec{r}}_{12} = \frac{1}{\mu}\vec{F}_{12}(\vec{r}_{12}) \end{cases}$$

or

$$\left[\begin{array}{l} \ddot{\vec{r}}_{cm} = \vec{g} \\ \mu\ddot{\vec{r}}_{12} = \vec{F}_{12}(\vec{r}_{12}) \end{array}\right.$$

← the "equivalent one-body problem"

Therefore, for an isolated two-body system or when only a uniform gravitational field is present, the two-body problem reduces to two uncoupled one-body problems in terms of $\vec{r}_{cm} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2$ and $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$

$$\begin{cases} \vec{r}_{cm} \text{ depends only on external grav. field} \\ \vec{r}_{12} \text{ depends only on internal force} \end{cases}$$

Solve the 2 one-body problems [first one is easy]

then find $\vec{r}_1$ and $\vec{r}_2$ by

inverting the eqns for $\vec{r}_{cm}$ + $\vec{r}_{12}$

$$\begin{cases} \vec{r}_1 = \vec{r}_{cm} + \frac{m_2}{M}\vec{r}_{12} \\ \vec{r}_2 = \vec{r}_{cm} - \frac{m_1}{M}\vec{r}_{12} \end{cases}$$

[verify these work!]

Thus the 2-body problem is solvable [under conditions stated]

The 3- (or N-) body problem is notoriously unsolvable

Momentum of 2-body system

Recall $M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2$, where $M = m_1 + m_2$

then $M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{p}^{\,sys}$

- momentum of the c-m = total momentum of the system

Also, the center of mass frame
(frame in which center of mass is not moving $\Rightarrow \vec{v}_{cm} = 0$)
is $\therefore$ the zero momentum frame $(\vec{p}^{\,sys} = 0)$

Since $\vec{F}^{ext} = M\ddot{\vec{r}}_{cm}$

$$= \frac{d}{dt}(M\vec{v}_{cm})$$

$$= \frac{d\vec{p}^{\,sys}}{dt}$$

The change in total mom of the system depends only on external forces.

Corollary: The total momentum of an isolated system is conserved

$$\vec{F}^{ext} = 0 \Rightarrow \vec{p}^{\,sys} = \text{const}$$

Kinetic energy of 2-body system

$$T^{sys} = T_1 + T_2$$

$$= \frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2}$$

$$\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$$

Using $\begin{cases} \vec{v}_1 = \vec{v}_{cm} + \frac{m_2}{M} \vec{v}_{12} \\ \vec{v}_2 = \vec{v}_{cm} - \frac{m_1}{M} \vec{v}_{12} \end{cases}$

one has

$$T^{sys} = \frac{1}{2} m_1 \left(\vec{v}_{cm} + \frac{m_2}{M} \vec{v}_{12}\right)^2 + \frac{1}{2} m_2 \left(\vec{v}_{cm} - \frac{m_1}{M} \vec{v}_{12}\right)^2$$

$$= \frac{1}{2}(m_1 + m_2) \vec{v}_{cm}^{\,2} + 0 + \frac{1}{2}\left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}\right) \vec{v}_{12}^{\,2}$$

$$= \frac{1}{2} M \vec{v}_{cm}^{\,2} + \underbrace{\frac{1}{2} \mu v_{12}^2}$$

Kinetic energy of equivalent 1-body problem

Also, kinetic energy of system in the cm frame

[Note: $\frac{m_2}{M}\dot{\vec{r}}$ & $\frac{-m_1}{M}\dot{\vec{r}}$
are velocities $\dot{\vec{r}}_1'$ and $\dot{\vec{r}}_2'$
in cm frame]

<u>change in kinetic energy</u>

$$dT^{sys} = dT_1 + dT_2$$

$$= (\vec{F}_{12} + \vec{F}_1^{ext}) \cdot d\vec{r}_1 + (-\vec{F}_{12} + \vec{F}_2^{ext}) \cdot d\vec{r}_2$$

$$= \vec{F}_{12} \cdot \underbrace{(d\vec{r}_1 - d\vec{r}_2)}_{d\vec{r}_{12}} + \underbrace{\vec{F}_1^{ext} \cdot d\vec{r}_1 + \vec{F}_2^{ext} \cdot d\vec{r}_2}_{\substack{\text{work done by external forces} \\ dW^{ext}}}$$

If internal force is constraint force [eg. rigid rod]

$$\vec{F}_{12} \cdot d\vec{r}_{12} = 0$$

If internal force is conservative

$$\vec{F}_{12} \cdot d\vec{r}_{12} = -dU_{12}$$

$$dT_1 + dT_2 + dU_{12} = dW^{ext}$$

Mechanical energy of a 2-body system

$$E_{mech} = T_1 + T_2 + U_{12}$$

$$dE_{mech} = dW^{ext}$$

Corollary: for an isolated system
(no external forces),
mechanical energy is conserved, $E_{mech} = \text{const}$

$$E_{mech} = T_1 + T_2 + U_{12}$$

$$= \underbrace{\tfrac{1}{2} M \vec{v}_{cm}^{\,2}}_{\text{kinetic energy of cm}} + \Big[\underbrace{\tfrac{1}{2} \mu \vec{v}_{12}^{\,2} + U_{12}(\vec{r}_{12})}_{\text{mech energy of equivalent 1-body problem}} \Big]$$

separately conserved

[Nonconservative internal forces, ⟹
mechanical energy can be converted to thermal
eg 2 colliding balls of putty.]

Consider an extreme asymmetric two body system in which $m_2 >> m_1$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \approx m_1$$ (μ = lighter mass)

$$\vec{r}_1 = \vec{r}_{cm} + \frac{m_2}{M}\vec{r}_{12}, \quad \vec{r}_2 = \vec{r}_{cm} - \frac{m_1}{M}\vec{r}_{12}$$

In cm frame ($\vec{r}_{cm} = 0$, $\vec{v}_{cm} = 0$)

$$\vec{r}_1 = \frac{m_2}{M}\vec{r}_{12} \approx \vec{r}_{12} \qquad \vec{r}_2 = -\frac{m_1}{M}\vec{r}_{12} \approx 0$$

$$\vec{v}_1 \approx \vec{v}_{12}$$

$$\text{so } E_{mech} \cong \underbrace{\tfrac{1}{2}M\vec{v}_{cm}^{\,2}}_{=0} + \underbrace{\left[\tfrac{1}{2}m_1 v_1^2 + U_{12}(r_{12})\right]}$$

Sometimes we speak of the potential energy of the object

Thus we can ignore the elephant (m_2) & treat the 2 body problem as just the 1-body problem of the first mass optional

Derive Newton's 2nd law for equivalent one-body problem
↳ call $\vec{r}_{12} \to \vec{r}$

$$\frac{1}{2}\mu\vec{v}^2 + U_{12}(\vec{r}) = \text{const}$$

$$\frac{d}{dt}\left(\frac{1}{2}\mu\vec{v}^2 + U_{12}(\vec{r})\right) = 0$$

$$\mu\vec{v}\cdot\vec{a} + \vec{\nabla}U_{12}\cdot\frac{d\vec{r}}{dt} = 0$$

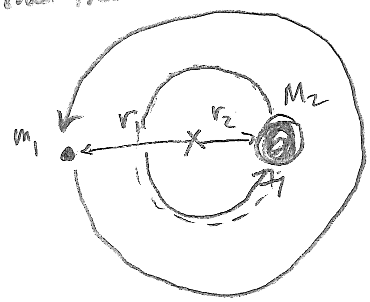
$$\mu\vec{a} + \underbrace{\vec{\nabla}U_{12}}_{-\vec{F}_{12}} = 0$$

$$\mu\vec{a} = \vec{F}_{12}$$

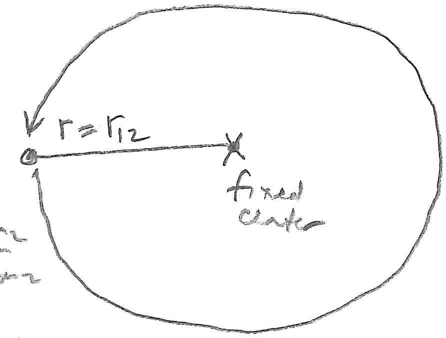
~~Consider~~

An isolated 2-body system is equivalent to a 1-body problem

E.g. consider a gravitationally-bound binary system
Both orbit around the cm



$=$ μ



$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\mu \vec{a} = \vec{F}_{12} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

Assume circular orbit: $\vec{a} = -\frac{v^2}{r}\hat{r} = -\omega^2 r \hat{r}$

$$\mu \omega^2 r = \frac{G m_1 m_2}{r^2}$$

$$\omega^2 r^3 = \frac{G m_1 m_2}{\mu} = G(m_1 + m_2)$$

$$\left(\frac{2\pi}{T}\right)^2 r^3 = G(m_1 + m_2)$$

Kepler's 3rd law
depends on <u>sum</u> of masses only

Sommerfeld Mechanics §1.3.5

mechanical energy of isolated 2-body system

$$E_{mech} = \frac{1}{2} M \vec{V}_{cm}^{\,2} + \left[\frac{1}{2} \mu \vec{v}_{12}^{\,2} + U_{12}(\vec{r}_{12}) \right]$$

Consider an elastic collision. Well before and well after collision

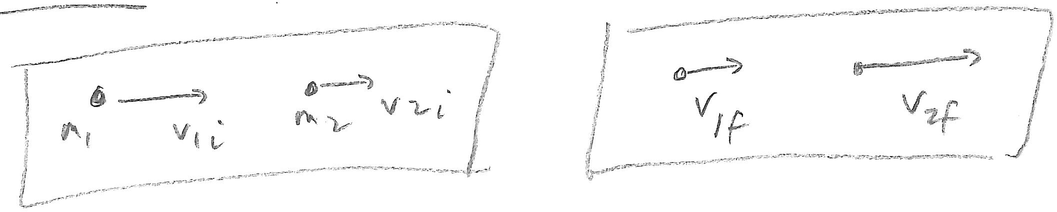
$U_{12} \to 0$, so $\quad E_{mech} = \frac{1}{2} M \vec{V}_{cm}^{\,2} + \frac{1}{2} \mu \vec{v}_{12}^{\,2}$

- $\vec{V}_{cm}$ is unchanged in collision [due to internal forces]
- Since E_{mech} conserved in elastic collision, $\vec{v}_{12}^{\,2}$ is unchanged

$$|\vec{v}_{rel,final}| = |\vec{v}_{rel,init}|$$

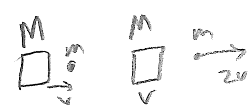
speed of recession = speed of approach

one dimension



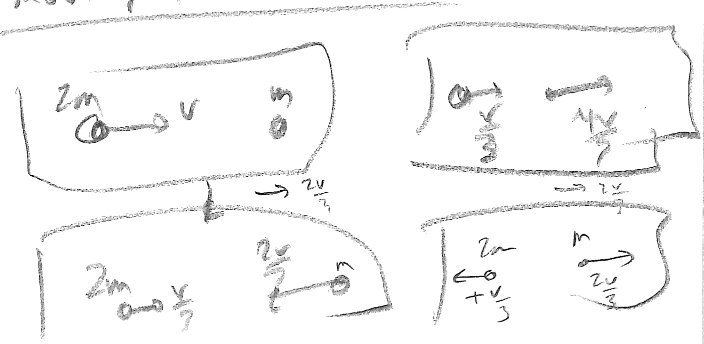
in cm frame

- equal masses
- very unequal: ping pong ball + stationary truck
  stationary ping pong ball + moving truck



Then

Do eg



inelastic collisions: internal forces are non conservative
so internal kinetic energy is lost to heat

$$\Rightarrow |\vec{v}_{rel,final}| < |\vec{v}_{rel,init}|$$

completely ~~extremely~~ inelastic: $\Rightarrow \quad |\vec{v}_{rel,final}| = 0$

objects stick together

Angular momentum

For a point mass, the angular momentum is

$$\vec{L} = \vec{r} \times \vec{p}$$

The torque exerted on the mass by a force $\vec{F}$ is

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Always keep in mind that both $\vec{L}$ & $\vec{\tau}$ depend on the choice of the origin from which $\vec{r}$ is defined.

Consider

$$\frac{d\vec{L}}{dt} = \underbrace{\dot{\vec{r}} \times \vec{p}}_{0} + \vec{r} \times \underbrace{\dot{\vec{p}}}_{\vec{F}}$$

because $p = m\dot{\vec{r}}$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

Angular momentum of the 2-body system is the vector sum

$$\vec{L}^{sys} = \vec{L}_1 + \vec{L}_2$$

$$\frac{d\vec{L}^{sys}}{dt} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

$$= \vec{r}_1 \times (\vec{F}_1^{ext} + \vec{F}_{12}) + \vec{r}_2 \times (\vec{F}_2^{ext} - \vec{F}_{12})$$

$$= \underbrace{\vec{\tau}_1^{ext} + \vec{\tau}_2^{ext}}_{\vec{\tau}^{ext}} + \vec{r}_{12} \times \vec{F}_{12}$$

If the internal force is a central force then $\vec{F}_{12} \parallel$ (anti)parallel to $\vec{r}_{12}$

so last term vanishes

$$\frac{d\vec{L}^{sys}}{dt} = \vec{\tau}^{ext}$$

change in ang. mom depends only on external torque

Corollary: isolated system $\Rightarrow$ $\vec{L}^{sys}$ is conserved.

$$\vec{L}^{sys} = \vec{L}_1 + \vec{L}_2$$

$$= m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2$$

$$= m_1 \left(\vec{r}_{cm} + \frac{m_2}{M}\vec{r}_{12}\right) \times \left(\vec{v}_{cm} + \frac{m_2}{M}\vec{v}_{12}\right)$$

$$+ m_2 \left(\vec{r}_{cm} - \frac{m_1}{M}\vec{r}_{12}\right) \times \left(\vec{v}_{cm} - \frac{m_1}{M}\vec{v}_{12}\right)$$

cross terms cancel

$$= (m_1 + m_2)(\vec{r}_{cm} \times \vec{v}_{cm}) + \underbrace{\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}}_{\frac{m_1 m_2}{m_1 + m_2} = \mu} \vec{r}_{12} \times \vec{v}_{12}$$

$$= \underbrace{\vec{r}_{cm} \times M\vec{v}_{cm}}_{\text{angular mom. of c.m.}} + \underbrace{\vec{r}_{12} \times \mu\vec{v}_{12}}_{\text{angular mom of equivalent 1-body problem}}$$

external torque
can change both of these