Eqns of motion for rigid body

$$\left(\frac{d\vec{L}}{dt}\right)_F = \vec{N} \quad \text{in } F = \text{fixed IRF}$$

$$\left(\frac{d\vec{L}}{dt}\right)_F = \left(\frac{d\vec{L}}{dt}\right)_{F'} + \vec{\omega}\times\vec{L} \qquad F' = \text{(rotating) body frame}$$

$$\boxed{\left(\frac{dL}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = \vec{N}}$$

Choose body axes to be principle axes $\hat{x}_1, \hat{x}_2, \hat{x}_3$

$$\vec{L} = L_i\,\hat{x}_i \qquad L_1 = I_1\omega_1 \text{ etc.}$$

$$\vec{\omega} = \omega_i\,\hat{x}_i$$

$$\frac{d\vec{L}}{dt} = \dot{L}_i\,\hat{x}_i = I_i\dot{\omega}_i\,\hat{x}_i$$

$$\vec{\omega}\times\vec{L} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$\left.\begin{aligned} N_1 &= I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 \\ N_2 &= I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 \\ N_3 &= I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 \end{aligned}\right\} \quad \text{Euler's eqns (in rotating body frame)}$$

Torque free motion $\quad \vec{N}_b = 0$

Start rotation about axis 1 (almost)

$\omega_1 >> \omega_2 ; \omega_3$

$$\frac{d\omega_1}{dt} = \frac{I_2 - I_3}{I_1} \omega_2 \omega_3 \approx 0 \quad \Rightarrow \omega_1 \approx \text{const}$$

$$\begin{cases} \dfrac{d\omega_2}{dt} = \left(\dfrac{I_3 - I_1}{I_2}\right) \omega_1 \omega_3 \\[2ex] \dfrac{d\omega_3}{dt} = \dfrac{I_1 - I_2}{I_3} \omega_1 \omega_2 \end{cases}$$

$$\frac{d^2\omega_2}{dt} \approx \frac{I_3 - I_1}{I_2} \omega_1 \frac{d\omega_3}{dt} = \left[\frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3} \omega_1^2 \right] \omega_2$$

If I_1 is max or min moment of inertia

$I_1 > I_3, I_2$ or $I_1 < I_3, I_2$, then

$[\quad] < 0$ and $\dfrac{d^2\omega_2}{dt} = -k^2 \omega_2$

hmc osc.

perturbations ~~on~~ in ω_2, ω_3 stable

If I_1 is intermediate axis, then $[\;] > 0$

$\dfrac{d^2\omega_2}{dt} = +k^2 \omega_2$ exponential growth

$\Rightarrow$ unstable

Torque free motion of a figure of revolution

$I_1 = I_2 = I_\perp$

Earth $I_3 \gtrsim I_\perp$

$$\frac{I_3 - I_\perp}{I_\perp} \approx \frac{1}{300}$$

$$\dot{\omega}_1 = \frac{I_\perp - I_3}{I_\perp}\,\omega_2\omega_3$$

$$\dot{\omega}_2 = -\frac{I_\perp - I_3}{I_\perp}\,\omega_1\omega_3$$

$$\dot{\omega}_3 = 0 \quad \Rightarrow \omega_3 = \text{const}$$

$$\ddot{\omega}_1 = \left(\frac{I_\perp - I_3}{I_\perp}\right)^2 \omega_3^2\,\omega_1$$

$$\Omega = \left(\frac{I_3 - I_\perp}{I_\perp}\right)\omega_3$$

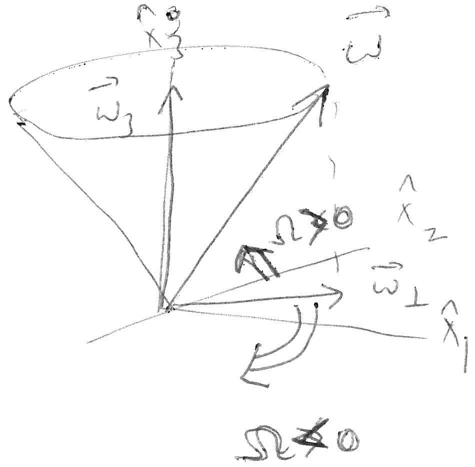
$$\omega_1 = A\cos(\Omega t)$$

$$\dot{\omega}_1 = -A\Omega\sin(\Omega t) = -\Omega\,\omega_2$$

for earth $\Omega > 0$

$$\Rightarrow \omega_2 = +A\sin(\Omega t)$$

$$\omega_\perp = \sqrt{\omega_1^2 + \omega_2^2} = A, \text{ const}$$



$\vec{\omega}$ precesses about $\hat{x}_3$

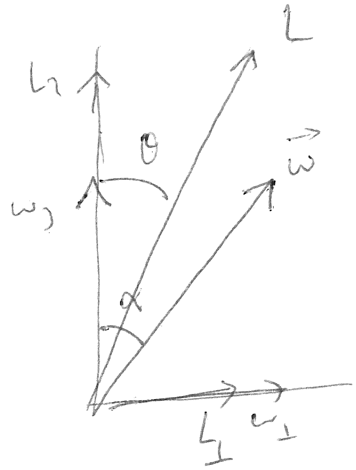
w/ freq Ω

Earth: $\Omega = \frac{\omega_3}{300}$

$\Rightarrow$ period = 300 days

(Chandler wobble ~400 days)

Angular momentum



$\omega_3 = \omega \cos\alpha$

$\omega_\perp = \omega \sin\alpha$

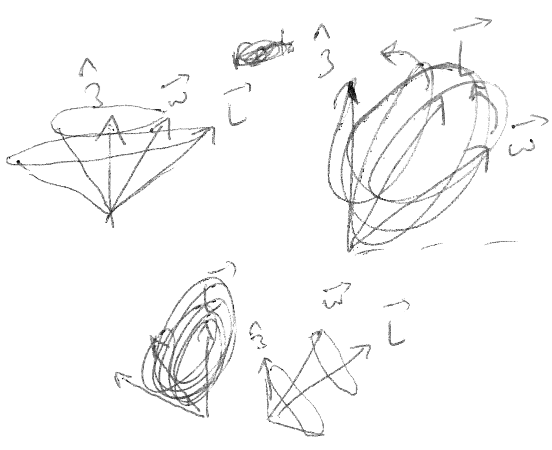
$L_3 = I_3 \omega_3 = I_3 \omega \cos\alpha$

$L_\perp = I_\perp \omega_\perp = I_\perp \omega \sin\alpha$

$\frac{L_\perp}{L_3} = \tan\theta = \frac{I_\perp}{I_3} \tan\alpha$

$I_3 > I_\perp \Rightarrow \theta < \alpha$ (earth)

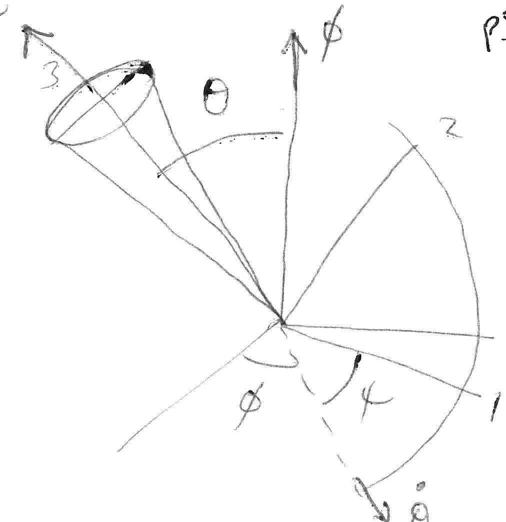
$I_3 < I_\perp \Rightarrow \theta > \alpha$



In rotating frame $\vec{\omega}$ & $\vec{L}$ precess about 3 w/ freq Ω.

In fixed frame $\vec{L}$ = const and $\vec{\omega}$ and $\hat{x}_3$ precess about it p5

<u>Top</u>



$$T = \frac{1}{2} I_\perp (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

$$V = + mgh \cos\theta$$

$\dot\phi$ = precession
$\dot\psi$ = spin
$\dot\theta$ = nutation

$$\left\{ \begin{aligned} \omega_3 &= \dot\psi + \dot\phi \cos\theta \\ \omega_1 &= \dot\theta \cos\psi + \dot\phi \sin\theta \sin\psi \\ \omega_2 &= \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \end{aligned} \right.$$

← go thru argument

$$\omega_1^2 + \omega_2^2 = \dot\theta^2 + \dot\phi^2 \sin^2\theta$$

$$L = \frac{1}{2} I_\perp (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2 - mgh\cos\theta$$

cyclic in ϕ and ψ

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_\perp \dot\phi \sin^2\theta + I_3 (\dot\phi \cos\theta + \dot\psi)\cos\theta = \text{const}$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 \underbrace{(\dot\phi\cos\theta + \dot\psi)}_{\omega_3} = \text{const}$$

$$p_\phi - p_\psi \cos\theta = I_\perp \dot\phi \sin^2\theta$$

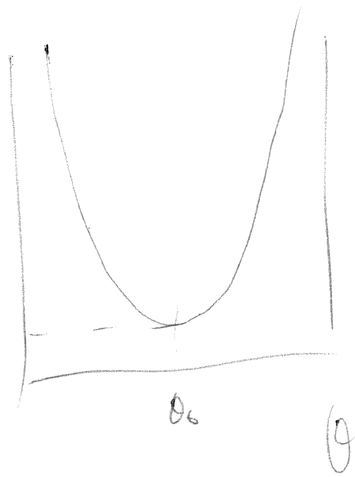
$$\dot\phi = \frac{p_\phi - p_\psi \cos\theta}{I_\perp \sin^2\theta}$$

$$E = \frac{1}{2} I_\perp \left(\dot\phi^2 \sin^2\theta + \dot\theta^2 \right) + \frac{1}{2} I_3 \Big(\underbrace{\dot\phi \cos\theta + \dot\psi}_{} \Big)^2 + mgh\cos\theta$$

$$\left(\frac{p_\psi}{I_3}\right)^2$$

$$= \frac{1}{2} I_\perp \left(\left[\frac{p_\phi - p_\psi \cos\theta}{I_\perp \sin^2\theta} \right]^2 \sin^2\theta + \dot\theta^2 + \frac{1}{2}\frac{p_\psi^2}{I_3} + mgh\cos\theta \right)$$

$$= \frac{1}{2} I_\perp \dot\theta^2 + \underbrace{\left[\frac{(p_\phi - p_\psi\cos\theta)^2}{2 I_\perp \sin^2\theta} + mgh\cos\theta \right]}_{V_{eff}(\theta)}$$



Consider $\theta = \theta_0 = \text{const}$

$$\frac{\partial V_{eff}}{\partial\theta} = 0 = \frac{(p_\phi - p_\psi\cos\theta)(p_\psi\sin\theta)}{I_\perp \sin^2\theta} - \frac{(p_\phi - p_\psi\cos\theta)^2\cos\theta}{I_\perp \sin^3\theta} - mgh\sin\theta \Big|_{\theta_0}$$

$$\beta = p_\phi - p_\psi\cos\theta_0$$

$$0 = \beta(p_\psi \sin^2\theta_0) - \beta^2(\cos\theta_0) - I_\perp mgh \sin^4\theta$$

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4mgh I_\perp \cos\theta_0}{p_\psi^2}}\right)$$

β real $\Rightarrow 1 - \dfrac{4mgh I_\perp \cos\theta_0}{p_\psi^2} > 0$

$I_3 \omega_3^2 = p_\psi^2 \geq 4mgh\, I_\perp \cos\theta_0$

$p_\psi = I_3 \omega_3$

$\omega_3 \geq \dfrac{2}{I_3}\sqrt{mgh\, I_\perp \cos\theta_0}$ Minimum spin for precession

Rate of precession: $\dot\phi = \dfrac{\beta}{I_\perp \sin^2\theta}$

2 roots for $\beta \Rightarrow$ 2 precession rates:

Suppose fast spinning top: ω_3 large $\Rightarrow p_\psi$ large

$\sqrt{1 - \dfrac{4mgh I_\perp \cos\theta_0}{p_\psi^2}} \approx 1 - \dfrac{2mgh I_\perp \cos\theta_0}{p_\psi^2}$

$\beta_+ \approx p_\psi \dfrac{\sin^2\theta_0}{\cos\theta_0} \Rightarrow \dot\phi = \dfrac{p_\psi}{I_\perp \cos\theta_0} = \dfrac{I_3 \omega_3}{I_\perp \cos\theta_0}$ fast prec

$\beta_- \approx \dfrac{mgh I_\perp \sin^2\theta_0}{p_\psi} \Rightarrow \dot\phi = \dfrac{mgh}{I_3 \omega_3}$ slow prec