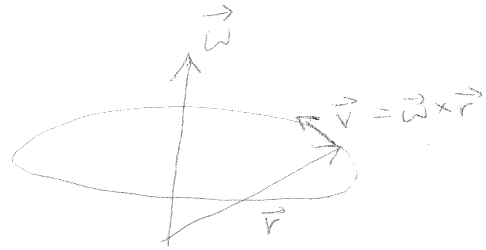
Angular momentum of a rigid body

$\hat{x}, \hat{y}, \hat{z}$ = body-fixed coords

$$\vec{\omega} = \omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}$$



$$\vec{L} = \sum m_n (\vec{r}_n \times \vec{v}_n)$$

$$\vec{v}_n = \vec{\omega} \times \vec{r}_n$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

(do ϵ_{ijk} stuff)
next page

$$\vec{L} = \sum m_n \left(r_n^2 \vec{\omega} - (\vec{r}_n \cdot \vec{\omega}) \vec{r}_n \right)$$

$$= \sum m_n \Big([x_n^2 + y_n^2 + z_n^2][\omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}] - [\omega_x x_n + \omega_y y_n + \omega_z z_n][x_n \hat{x} + y_n \hat{y} + z_n \hat{z}] \Big)$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \sum m_n \left([x_n^2 + y_n^2 + z_n^2]\omega_x - x_n^2 \omega_x - x_n y_n \omega_y - x_n z_n \omega_z \right) \\ \sum m_n \left([x_n^2 + y_n^2 + z_n^2]\omega_y - x_n y_n \omega_x - y_n^2 \omega_y - z_n y_n \omega_z \right) \\ \sum m_n \left([x_n^2 + y_n^2 + z_n^2]\omega_z - x_n z_n \omega_x - z_n y_n \omega_y - z_n^2 \omega_z \right) \end{pmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\begin{pmatrix} \sum m_n (y_n^2 + z_n^2) & -\sum m_n x_n y_n & -\sum m_n x_n z_n \\ -\sum m_n x_n y_n & \sum m_n (x_n^2 + z_n^2) & -\sum m_n y_n z_n \\ -\sum m_n x_n z_n & -\sum m_n y_n z_n & \sum m_n (y_n^2 + x_n^2) \end{pmatrix}}_{I} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$I_{ij} = \sum_n m_n \left[\delta_{ij} \left(\sum_k r_{n,k}^2 \right) - r_{n,i} r_{n,j} \right]$$

Continuum

$$I_{zz} = \sum m_n (x_n^2 + y_n^2) \longrightarrow \int dm \, (x^2 + y^2)$$

$\vec{d} = \vec{b} \times \vec{c}$

$d_i = \epsilon_{ijk} b_j c_k$

$\epsilon_{iik} = 0$ if $i = j, \ldots$

$\epsilon_{123} = \epsilon_{231} = \epsilon_{321} = 1$

$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \vec{d} = a_i d_i = \epsilon_{ijk} a_i b_j c_k$

$\vec{f} = \vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times \vec{d}$

$f_i = \epsilon_{ijk} a_j d_k = \epsilon_{ijk} \epsilon_{klm} a_j b_l c_m$

$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

$f_i = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m$

$= b_i (a \cdot c) - c_i (\vec{a} \cdot \vec{b})$

$\vec{f} = b(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$L_i = I_{ij}\,\omega_j$$

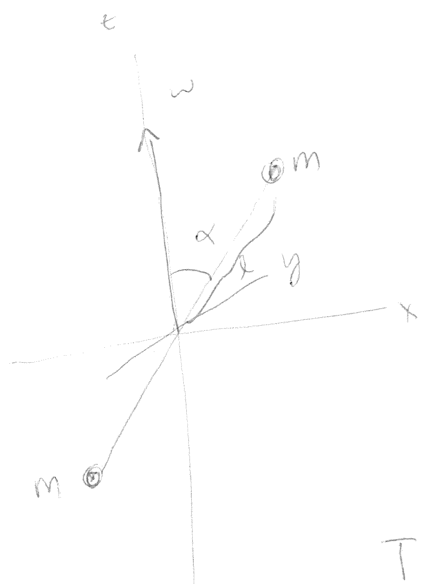
Inertia tensor: symmetric matrix

diag elts I_{xx} = moments of inertia

off diag elts I_{xy} = products of inertia

$\boxed{\text{reduces to usual } I_{zz} = \sum m_i (x^2+y^2) = \sum m_i r^2}$

$\vec{L}$ not necessarily $\parallel$ to $\vec{\omega}$



$x = \pm l \sin\alpha$

$y = 0$

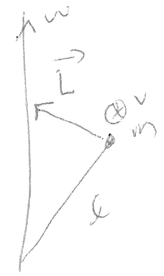
$z = \pm l \cos\alpha$

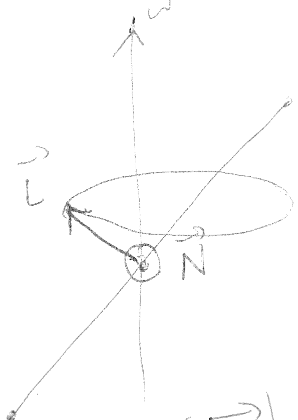
$$\underline{I} = \begin{pmatrix} 2ml^2\cos^2\alpha & 0 & -2ml^2\sin\alpha\cos\alpha \\ 0 & 2ml^2 & 0 \\ -2ml^2\sin\alpha\cos\alpha & 0 & 2ml^2\sin^2\alpha \end{pmatrix}$$

03

Let $\omega = \omega_z \hat{z} = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$

$\vec{L} = I\omega = \begin{pmatrix} -2ml^2 \sin\alpha\cos\alpha \\ 0 \\ 2ml^2 \sin^2\alpha \end{pmatrix}$





If released what would happen?

$\vec{\tau} = $ const ✓

$\left(\frac{d\vec{L}}{dt}\right)_{SRF} = \left(\frac{dL}{dt}\right)_{body} + \left(\vec{\omega} \times \vec{L}\right)$

$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega_z \\ -2ml^2 \sin\alpha\cos\alpha & 0 & 2ml^2 \sin^2\alpha \end{vmatrix}$

$= \begin{pmatrix} 0 \\ -2ml^2\omega^2 \sin\alpha\cos\alpha \\ 0 \end{pmatrix} = \vec{N}$

Kinetic energy of a rotating rigid body

$$T = \sum \frac{1}{2} m_i \vec{v}_i^{\,2}$$

$$= \sum \frac{1}{2} m_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i) \longrightarrow \epsilon_{ijk}$$

$$= \sum \frac{1}{2} m_i \left[\vec{\omega}^2 \vec{r}_i^{\,2} - (\vec{\omega} \cdot \vec{r}_i)^2 \right]$$

$$= \frac{\vec{\omega}}{2} \cdot \underbrace{\sum \cancel{} m_i \left[\vec{\omega}\, r_i^2 - \vec{r}_i (\vec{\omega} \cdot \vec{r}_i) \right]}_{\vec{L}}$$

$$= \frac{1}{2} \vec{\omega} \cdot I \omega$$

$$\frac{1}{2} (\omega_1 \ \omega_2 \ \omega_3) \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$= \frac{1}{2} \omega_i I_{ij} \omega_j$$

$$T_{total} = \frac{1}{2} m \vec{V}_{cm}^{\,2} + \frac{1}{2} \omega_i I_{ij} \omega_j$$

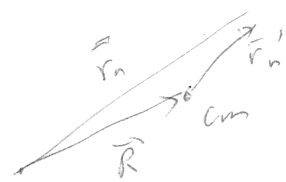
05

Let I_{ij} be the inertia tensor rel to ~~the~~ some orig O

Let I'_{ij} be the inertia tensor rel to cm

$$I_{ij} = \sum m_n \left[\delta_{ij} \left(\sum_k r_{n,k}^2 \right) - r_{n,i} \, r_{n,j} \right]$$

$$\vec{r}_n = \vec{R} + \vec{r}\,'_n$$



$$I_{ij} = \sum m_n \left[\delta_{ij} \left(\sum_k (R_k + r'_{n,k})^2 \right) - (R_i + r'_{n,i})(R_j + r'_{n,j}) \right]$$

$$= \underbrace{\sum m_n \left[\delta_{ij} \left(\sum_k r'_{n,k} \right)^2 - r'_{n,i} r'_{n,j} \right]}_{I'_{ij}}$$

$$+ \underbrace{\sum m_n}_{M} \left[\delta_{ij} \sum R_k^2 - R_i R_j \right]$$

$$+ \underbrace{\sum m_n \left[\delta_{ij} \sum_k 2 R_k r'_{n,k} - R_i r'_{n,j} - R_j r'_{n,i} \right]}_{0 \text{ because } \sum m_n r'_{n,l} = 0}$$

$$\boxed{I_{ij} = I'_{ij} + M \left[R^2 \delta_{ij} - R_i R_j \right]}$$

Steiner's parallel axis th

Special case

$$I_{zz} = I'_{zz} + M^2 [\underbrace{x^2 + y^2}_{}]$$

$r_\perp^2 \Rightarrow$ distance btw axes

$L'_a = A_{ai} L_i$ under transformation rotation

$\omega'_b = A_{bj} \omega_j \quad \Rightarrow \omega_j = \tilde{A}_{jb} \omega'_b$

$I'_{ab} = ??$

Relationships between physical quantities invariant

$L_i = I_{ij} \omega_j$

$L'_a = I'_{ab} \omega'_b$

$= A_{ai} L_i = A_{ai} I_{ij} \omega_j = A_{ai} I_{ij} \tilde{A}_{jb} \omega'_b$

$\Rightarrow I'_{ab} = A_{ai} I_{ij} \tilde{A}_{jb}$

$I' = A \, I \, \tilde{A} = A \, I \, A^{-1}$

$\Rightarrow I'_{ab} = A_{ai} A_{bj} I_{ij}$ tensor of 2nd rank

vector = tensor of 1st rank

$L'_a = A_{ai} L_i$

$B'_{abc} = A_{ai} A_{bj} A_{ck} B_{ijk}$, tensor of 3rd rank

$\vdots$

Scalar = 0th rank tensor

$L_i = I_{ij}\,\omega_j$

In general, $\vec{L}$ not $\parallel$ to $\vec{\omega}$.

Suppose $\vec{L} \parallel \vec{\omega}$, then $\vec{L} = \lambda\vec{\omega}$

$I_{ij}\,\omega_j = \lambda\,\omega_i$

$\underset{\text{matrix}}{\underset{\uparrow}{I}}\ \underset{\text{col}}{\underset{\uparrow}{\omega}} = \lambda\,\underset{\text{col}}{\underset{\uparrow}{\omega}}$ (eigenvalue eqn)

I is symmetric ($\therefore$ hermitian)

$\Rightarrow$ 3 real eigenvalues, called I_1, I_2, I_3 (principle ~~ex~~ moments of inertia)

+ 3 eigenvectors $\vec{\omega}_{(1)}, \vec{\omega}_{(2)}, \vec{\omega}_{(3)}$ (principle axes of inertia)

$\exists$ always 3 directions along which $\vec{L}$ is $\parallel$ to $\vec{\omega}$

$\vec{L}_{(1)} = I_1\,\vec{\omega}_{(1)}$

$\vec{L}_{(2)} = I_2\,\vec{\omega}_{(2)}$

$\vec{L}_3 = I_3\,\vec{\omega}_{(3)}$

~~To show: $\vec{\omega}_{(i)}$ are orthogonal~~

To show: $\hat{\omega}_{(n)}$ are $\perp$ to one another.

$$\mathbb{I}\, \omega_{(n)} = I_n\, \omega_{(n)}$$

$$\omega_{(m)}^T\, \mathbb{I}\, \omega_{(n)} = I_n\, \omega_{(m)}^T\, \omega_{(n)}$$

Take transpose

$$\omega_{(n)}^T\, \mathbb{I}^T\, \omega_{(m)} = I_n\, \omega_{(n)}^T\, \omega_{(m)}$$

$$= \omega_{(n)}^T\, \mathbb{I}\, \omega_{(m)}$$

$$= \omega_{(n)}^T\, I_m\, \omega_{(m)}$$

$$(I_n - I_m)\, \omega_{(n)}^T\, \omega_{(m)} = 0$$

$$I_n \neq I_m \Rightarrow \omega_{(n)}^T\, \omega_{(m)} = 0 \qquad \perp \checkmark$$

Principle axes are orthogonal.

Choose them as coordinate axes.

What is A?

$$A = \begin{pmatrix} \hat{\omega}_{(1)1} & \hat{\omega}_{(1)2} & \hat{\omega}_{(1)3} \\ \hat{\omega}_{(2)1} & \hat{\omega}_{(2)2} & \hat{\omega}_{(2)3} \\ \hat{\omega}_{(3)1} & \hat{\omega}_{(3)2} & \hat{\omega}_{(3)3} \end{pmatrix} = \begin{pmatrix} \hat{\omega}_{(1)}^T \\ \hat{\omega}_{(2)}^T \\ \hat{\omega}_{(3)}^T \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} \hat{\omega}_{(1)} & \hat{\omega}_{(2)} & \hat{\omega}_{(3)} \end{pmatrix}$$

$$I\tilde{A} = \begin{pmatrix} I_1 \hat{\omega}_{(1)} & I_2 \hat{\omega}_{(2)} & I_3 \hat{\omega}_{(3)} \end{pmatrix}$$

$$I' = A I \tilde{A} = \begin{pmatrix} \hat{\omega}_{(1)}^T \\ \hat{\omega}_{(2)}^T \\ \hat{\omega}_{(3)}^T \end{pmatrix} \begin{pmatrix} I_1 \hat{\omega}_{(1)} & I_2 \hat{\omega}_{(2)} & I_3 \hat{\omega}_{(3)} \end{pmatrix}$$

$$= \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

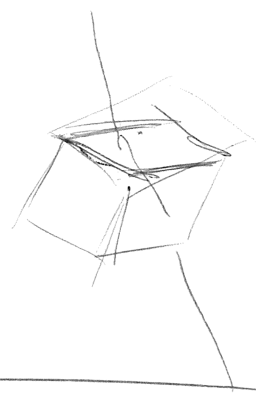
A symmetric mtx ~~is diag.~~ can be diagonalized by an orthogonal tx.

(A hermitian ~~mtx~~ can be diag. by a unitary tx.)

sphere : $I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_1 \end{pmatrix}$

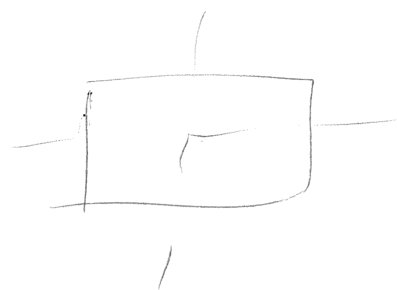
$I \to A I \tilde{A} = I$ so any axes are principle axes

cube : all axes are principle axes



dynamics stability

Planar figure

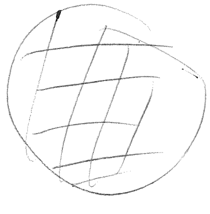


$I_{xx} = \sum m_n y_n^2$

$I_{yy} = \sum m_n x_n^2$

$I_{zz} = \sum m_n (x_n^2 + y_n^2) = I_{xx} + I_{yy}$

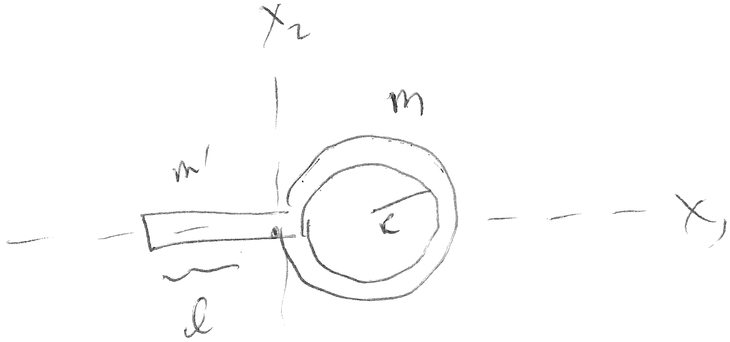
eg



disc $I_{zz} = \frac{1}{2} m R^2$

$\Rightarrow I_{xx} = \frac{1}{4} m R^2$

Tennis racket



$$I_1 = \frac{1}{2} m R^2$$

$$I_2 = \frac{1}{2} m R^2 + m R^2 + \frac{1}{3} m' \ell^2 = \frac{3}{2} m R^2 + \frac{1}{3} m' \ell^2$$

$$I_3 = m R^2 + m R^2 + \frac{1}{3} m' \ell^2 = 2 m R^2 + \frac{1}{3} m' \ell^2$$

$$= (I_1 + I_2)$$

I_2 = intermed. axis