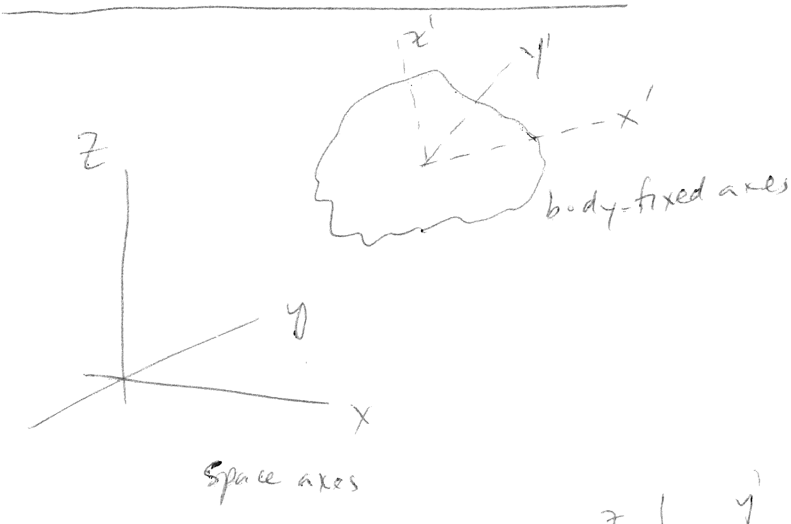
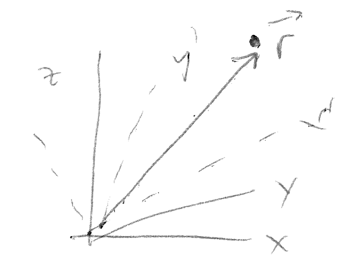
Orientation of a rigid body



Let origins coincide



$$\vec{r} = x_1 \hat{n}_1 + x_2 \hat{n}_2 + x_3 \hat{n}_3 = x_i \hat{n}_i, \quad \hat{n}_i = \text{space unit vectors}$$

$$= x_1' \hat{n}_1' + x_2' \hat{n}_2' + x_3' \hat{n}_3' = x_i' \hat{n}_i', \quad \hat{n}_i' = \text{body fixed unit vectors}$$

$$\hat{n}_i \cdot \hat{n}_j = \delta_{ij}, \quad \hat{n}_i' \cdot \hat{n}_j' = \delta_{ij}$$

x_i and x_i' are linearly related:

$$x_i' = A_{ij} x_j$$

$$x_1' = A_{11} x_1 + A_{12} x_2 + A_{13} x_3$$
$$x_2' = A_{21} x_1 + A_{22} x_2 + A_{23} x_3$$
$$x_3' = A_{31} x_1 + A_{32} x_2 + A_{33} x_3$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}}_{\left\{\begin{smallmatrix}\text{rotation} \\ \text{transformation}\end{smallmatrix}\right\} \text{ matrix.}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \mathbf{x}' = A\mathbf{x} \quad \leftarrow \text{matrix eqn}$$

N2

Consider

$$\hat{n}_1' \cdot \vec{r} = x_1' = \underbrace{\hat{n}_1' \cdot \hat{n}_1}_{A_{11}} x_1 + \underbrace{\hat{n}_1' \cdot \hat{n}_2}_{A_{12}} x_2 + \underbrace{\hat{n}_1' \cdot \hat{n}_3}_{A_{13}} x_3$$

$$A_{ij} = \hat{n}_i' \cdot \hat{n}_j = \cos(\hat{n}_i', \hat{n}_j)$$

direction cosines.

Any vector $\vec{V} = v_1 \hat{n}_1 + v_2 \hat{n}_2 + v_3 \hat{n}_3$

$$= v_1' \hat{n}_1' + v_2' \hat{n}_2' + v_3' \hat{n}_3'$$

$$v_i' = A_{ij} V_j$$

take this as definition of a vector

Nine elts of A_{ij} not all independent!
Six relations among them. Three indep parameters

$$X = B X'$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

$$X = BAX \quad \Rightarrow \quad BA = 1, \quad \boxed{B = A^{-1}}$$

$$\hat{n}_1 \cdot \vec{r} = x_1 = \underbrace{\hat{n}_1 \cdot \hat{n}_1'}_{B_{11}} x_1' + \underbrace{\hat{n}_1 \cdot \hat{n}_2'}_{B_{12}} x_2' + \underbrace{\hat{n}_1 \cdot \hat{n}_3'}_{B_{13}} x_3'$$

$$B_{ij} = \hat{n}_i \cdot \hat{n}_j' = \hat{n}_j' \cdot \hat{n}_i = A_{ji}$$

$$\boxed{B = \tilde{A}} = A^T$$

$$X = \tilde{A} X' = \tilde{A} A X \qquad \boxed{\tilde{A} A = 1}$$

If $A^{-1} = \tilde{A}$ ie $\tilde{A} A = 1$, then A is called

an orthogonal matrix;

Set (group) of such matrices is called $O(3)$

Orthogonality imposes six conditions

$$\cancel{\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & \end{pmatrix}} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}^2 + A_{12}^2 + A_{13}^2 & A_{11}A_{12} + A_{21}A_{22} + A_{31}A_{32} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A_{11}^2 + A_{12}^2 + A_{13}^2 = 1$ (~~each row is~~ sum of squares of each row (column) = 1)

$(A_{11}, A_{21}, A_{31}) \cdot (A_{12}, A_{22}, A_{32}) = 0$ (dot product of rows = 0)

$\boxed{\text{why?}}$: $\hat{n}_1 = (1, 0, 0)$ $x_1' = A_{11}$, $x_2' = A_{21}$, $x_3' = A_{31}$

Then by $x' = Ax$,

$\hat{n}_1 = A_{11}\hat{n}_1' + A_{21}\hat{n}_2' + A_{31}\hat{n}_3'$

so $\hat{n}_1 = $

$\hat{n}_1^2 = 1 \Rightarrow A_{11}^2 + A_{21}^2 + A_{31}^2 = 1$ ✓

$\hat{n}_2 = A_{12}\hat{n}_1' + A_{22}\hat{n}_2' + A_{32}\hat{n}_3'$

$\hat{n}_1 \cdot \hat{n}_2 = (A_{11}, A_{21}, A_{31})(A_{12}, A_{22}, A_{32}) = 0$ ✓

$\hat{n}_i = A_{ji}\hat{n}_i'$

$\cancel{\hat{n}_i \cdot \hat{n}_i = A_{ji} A_{ki} \hat{n}_j \cdot \hat{n}_k = A_{ji} A_{ji} = A_{ij} A_{ji} = 1}$ no sum on i

$\delta_{ik} = \hat{n}_i \cdot \hat{n}_k = A_{ji} A_{\ell k} \hat{n}_j \cdot \hat{n}_\ell = A_{ji} A_{jk} = \tilde{A}_{ij} A_{jk} =$

$\Rightarrow \tilde{A} A = 1$

~~Force-free motion~~ ~~(stable, unstable)~~ NS

<u>Euler's angles</u> : 3 angles that specify orientation

(ϕ, θ, ψ)

(refer to figure 10-7 of Marion)

Also, ~~work out~~ show how $A = A_\psi A_\theta A_\phi$

write ~~the~~ explicit form of A_ψ, A_θ, A_ϕ

~~$\tilde{A}A = 1 \Rightarrow A$ orthogonal~~

~~By definition $A^{-1}A = 1$ so $\tilde{A} = A^{-1} \Rightarrow A$ orthogonal~~

~~$\tilde{A} = A \Rightarrow A$ symmetric~~

~~Set (group) of such matrices called $O(3)$~~ ← orthogonal

~~→ 6 constraints, 9 vars, ⇒ 3 degrees of freedom~~

Orthogonal matrices have det $= \pm 1$.

$$\det(\tilde{A}A) = \underbrace{(\det \tilde{A})}_{\det A}(\det A) = (\det 1) = 1$$

$$(\det A)^2 = 1 \Rightarrow \det A = \pm 1$$

$SO(3)$ = Orthogonal matrices w/ det $= 1$

↑ "special"

Proper rotations have det 1 = { continuity from 1 }

det -1 ⇒ inversions $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ → rh coords → lh coords

↓ improper rotations

$x' = -x$
$y' = -y$
$z' = -z$

polar vectors $\vec{r}\,' = A\vec{r}$
axial vectors $\vec{r}\,' = (\det A)A\vec{r}$
→ same under rots, opp under inv.

Complex $\quad \psi' = U\psi \quad U^{\dagger} = (U^T)^*$ ← adjoint

$U^{\dagger}U = 1 \Rightarrow U$ unitary $\quad U(3)$, $SU(3)$

$U^{\dagger} = U$ hermitian

N7

Length of a vector is invariant under rotations

$$\vec{r}^2 = x_1^2 + x_2^2 + x_3^2 = x_i x_i = (x_1\ x_2\ x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x' = A x \qquad x = A^T x'$$

$$x'^T = x^T A^T \qquad x^T = x'^T A$$

$$\vec{r}^2 = x^T x = x'^T \underbrace{A A^T}_{1} x' = x'^T x' = x'_i x'_i = \vec{r}'^2$$

Special relativity:

Interval $(\Delta x)^2 = x_1^2 + x_2^2 + x_3^2 - t^2$

$$= (x_1\ x_2\ x_3\ t)\underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}}_{\eta}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ t \end{pmatrix}$$

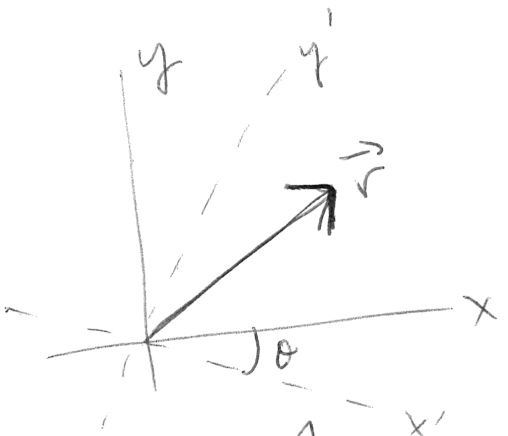
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ t' \end{pmatrix} = \Lambda \Big(x$$

$$\Lambda \eta \Lambda^T = \eta \Rightarrow SO(3,1)$$

Skip

Active vs passive trcs

2d: matrices 4 d.o.f.
$\underbrace{\hat{i}^2=\hat{j}^2=1,\ \hat{i}\cdot\hat{j}=0}_{\text{constraints}} \Rightarrow 1$ d.o.f.



clockwise rot of coord sys

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(\vec{r})' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

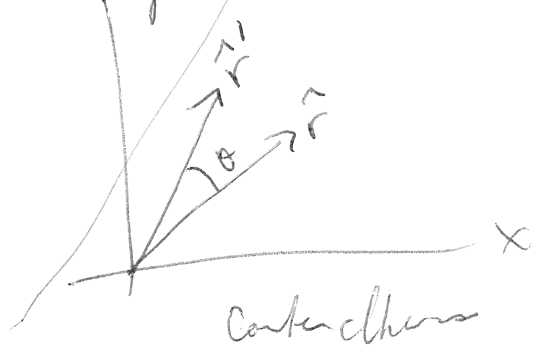
$$(\vec{r})' = A\,\vec{r}$$

same vector in diff coords

(passive)

Active tr: A rotates $\vec{r}$ to give a new vector $\vec{r}\,'$ in same coord sys

$$\vec{r}\,' = A\vec{r}$$



Counterclockwise rotat of vector ...

~~Skip~~

(Q: do finite rotats formula?)

rotat = matrix

infinitesimal rotat can be specified by a vector

ie axis of infinitesimal rotat...

$$A = 1 + \underset{\uparrow \text{small}}{\epsilon}$$

$$A^T = 1 + \epsilon^T$$

$$A^T A = (1+\epsilon^T)(1+\epsilon) = 1 + \epsilon^T + \epsilon + O(\epsilon^2) = 1$$

$$\epsilon^T = -\epsilon \quad \Rightarrow \text{antisymmetric}$$

$$\epsilon = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix}$$

$$r' = A r = r + \epsilon r$$

$$d\vec{r} = \vec{r}' - \vec{r} = \epsilon r = \begin{pmatrix} x_2 d\Omega_3 - x_3 d\Omega_2 \\ x_3 d\Omega_1 - x_1 d\Omega_3 \\ x_1 d\Omega_2 - x_2 d\Omega_1 \end{pmatrix}$$

$$= (\vec{r} \times d\vec{\Omega})$$

$$d\vec{\Omega} = \hat{n}\, d\theta$$

$$d\vec{x} = \hat{n} \times \hat{r}\, d\theta$$

But eg $AB \neq BA$; if rotats = vectors, then cd add them $\Rightarrow$ commute

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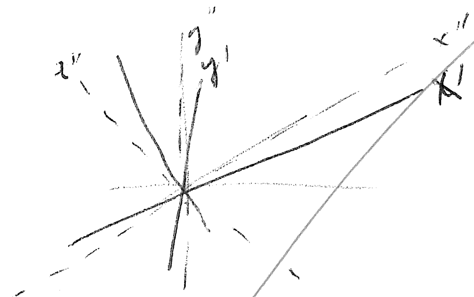
time t: $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A(t) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = A(t+dt) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\Delta A)\, A(t) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Delta A \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$\Delta A = 1 + \epsilon \ \delta t$

$d\vec{r} = \vec{r} \times \hat{n}\, d\theta$

$\dfrac{d\vec{r}}{dt} = \vec{r} \times \hat{n}\, \omega$

$\hat{n}$ = instantaneous axis of rotation