Two-body central force problem

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

Reduction to one-body problem

Change coordinates

$$\begin{cases} \text{center of mass} & \vec{R} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \\ \text{relative displacement} & \vec{r} = \vec{r}_1 - \vec{r}_2 \end{cases}$$

$$\left.\begin{aligned} \vec{r}_1 &= \vec{R} + \vec{r}_1{}' \\ \vec{r}_2 &= \vec{R} + \vec{r}_2{}' \end{aligned}\right\} \quad m_1 \vec{r}_1{}' + m_2 \vec{r}_2{}' = 0$$

$$L = \frac{1}{2}(m_1 + m_2) \dot{\vec{R}}^2 + \frac{1}{2} m_1 \dot{\vec{r}}_1{}'^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2{}'^2 - U(r)$$

$$r = |\vec{r}|$$

$$\vec{r}_2{}' = -\frac{m_1}{m_2} \vec{r}_1{}'$$

$$\vec{r} = \vec{r}_1{}' - \vec{r}_2{}' = \left(1 + \frac{m_1}{m_2}\right) \vec{r}_1{}' = \frac{m_1 + m_2}{m_2} \vec{r}_1{}'$$

$$\vec{r}_1{}' = \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}_2{}' = -\frac{m_1}{m_1 + m_2} \vec{r}$$

$$L = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \underbrace{\frac{1}{2}m_1 \frac{m_2^2}{(m_1+m_2)^2}\dot{\vec{r}}^2 + \frac{1}{2}m_2 \frac{m_1^2}{(m_1+m_2)^2}\dot{\vec{r}}^2}_{\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2} - U(r)$$

$$\frac{1}{2}\underbrace{\frac{m_1 m_2}{m_1+m_2}}_{\mu\text{, reduced mass}}\dot{\vec{r}}^2$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

[notice that]

$$\mu \leq m_1, m_2$$

$$m_1 \to \infty \Rightarrow \mu \to m_2$$

$$\frac{\partial L}{\partial \vec{R}} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{R}}}\right) = \frac{d}{dt}\left((m_1+m_2)\dot{\vec{R}}\right) = \frac{d\vec{P}}{dt} = 0$$

$$\vec{P} = \text{const}, \quad \vec{R} = \vec{R}_0 + \vec{v}_{cm}t$$

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$ ← one body problem in external potential

Solve for $\vec{r}$, then
$$\begin{cases} \vec{r}_1 = \vec{R} + \dfrac{m_2}{m_1+m_2}\vec{r} \\ \vec{r}_2 = \vec{R} - \dfrac{m_1}{m_1+m_2}\vec{r} \end{cases}$$

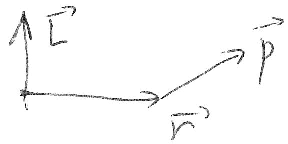
$$L = \frac{1}{2} m \dot{\vec{r}}^{\,2} - U(r)$$

[$\mu \rightarrow m$ note]

spherical symmetry $\Rightarrow \vec{L}$ conserved

$$\vec{F} = -\vec{\nabla} U = -\left(\frac{\partial U}{\partial r}\right)\hat{r}$$

$$\vec{N} = \vec{r} \times \vec{F} = 0 \Rightarrow \vec{L} \text{ conserved}$$



Reduction to two dimensions

$$\vec{L} = \vec{r} \times \vec{p}$$

$\vec{r}, \vec{p} \perp \vec{L}$

$\vec{r}, \vec{p}$ remain in plane $\perp$ to $\vec{L}$

Choose z axis along $\vec{L} \Rightarrow \vec{r}, \vec{p}$ in xy plane

Polar ~~coordinates in xy~~

$$\cancel{L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)}$$

$$\cancel{\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0}$$

$\cancel{mr^2\dot{\theta} = l}$ ~~orbital angular momentum~~

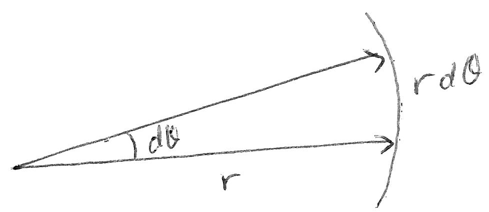
$$\cancel{\dot{\theta} = \frac{l}{mr^2}}$$

Polar coordinates

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - U(r)$$

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m r^2 \dot{\theta}\right) = 0$$

$m r^2 \dot{\theta} = \text{const} = \ell$, orbital angular momentum



$$dA = \frac{1}{2} r (r\, d\theta)$$

areal velocity $\dfrac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \dfrac{\ell}{2m} = \text{const}$

Kepler's Second Law = equal areas swept out in equal time

$\equiv$

Conservation of angular momentum

$$\frac{\partial L}{\partial \dot r} = m\dot r$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{\partial L}{\partial r}$$

$$m\ddot r = mr\dot\theta^2 - \frac{dU}{dr}$$

$$\dot\theta = \frac{\ell}{mr^2}$$

$$F = -\frac{dU}{dr}$$

$$m\ddot r = \underbrace{\frac{\ell^2}{mr^3}}_{F_c} + F$$

"centrifugal force"

$$F_c = -\frac{dU_c}{dr}$$

$$U_c = \frac{\ell^2}{2mr^2}$$

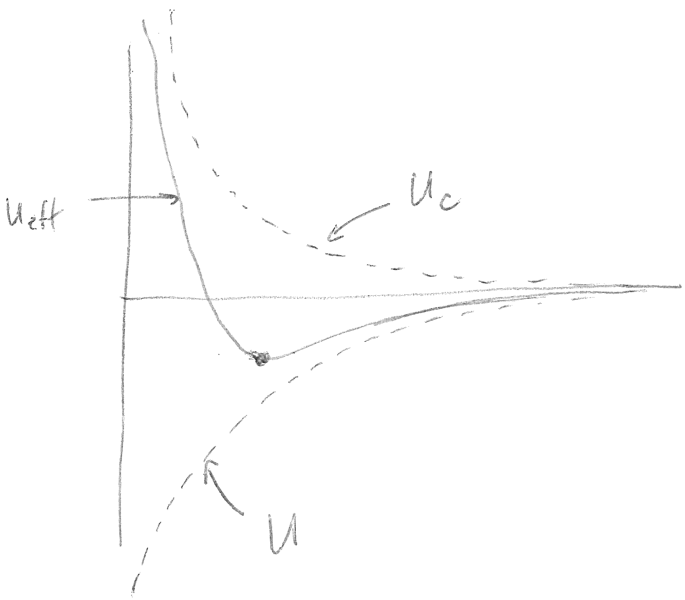
$$m\ddot r = -\frac{d}{dr}(U_c + U)$$

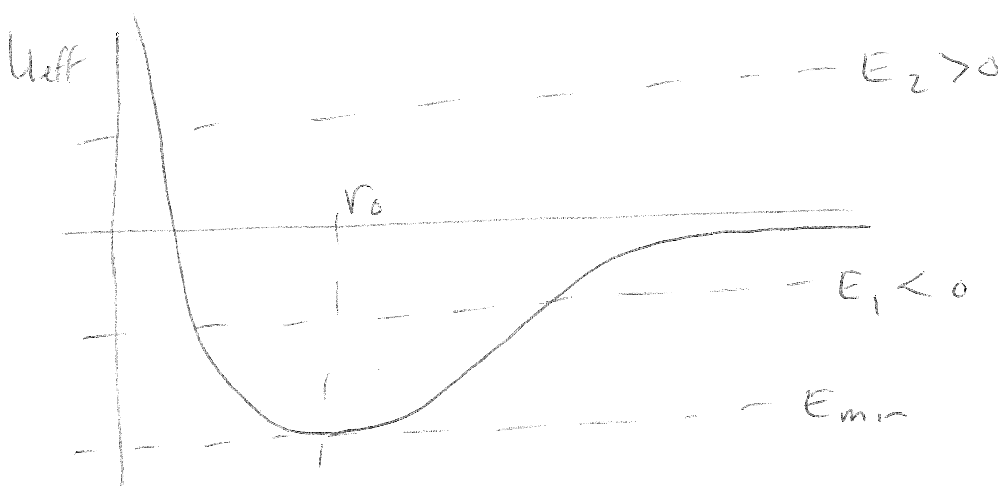
$$\underbrace{m\dot r\ddot r}_{\frac{d}{dt}\left(\frac{1}{2}m\dot r^2\right)} = -\frac{d}{dr}(U_c+U)\frac{dr}{dt} = -\frac{d}{dt}(U_c+U)$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot r^2 + U + U_c\right) = 0$$

$$\frac{1}{2}m\dot r^2 + \underbrace{U(r) + U_c(r)}_{U_{eff}(r)} = E$$

$U(x)$ attractive

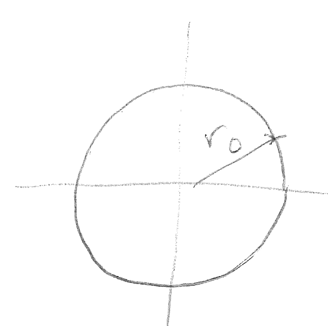




① $E = E_{min}$

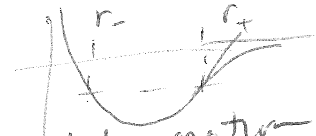
$$\frac{dU_{eff}}{dr} = 0 = \cancel{-\frac{l^2}{mr^3} + \frac{dU}{dr}}$$

$$F_{eff} = 0 = F + \frac{l^2}{mr^3}$$



$$\left.\begin{aligned} r &= r_0 = \text{const} \\ \dot{\theta} &= \frac{l}{mr_0^2} \\ \theta &= \frac{l}{mr_0^2}t + \theta_0 \end{aligned}\right\} \text{circular motion}$$

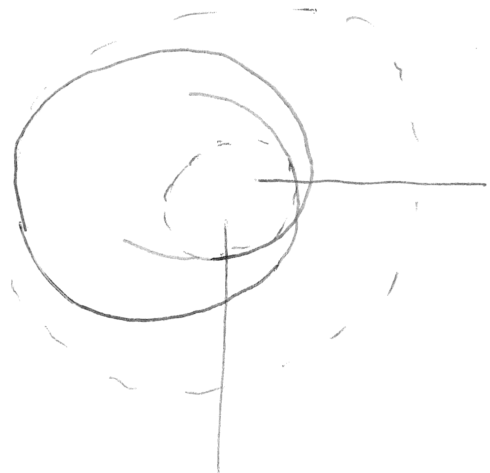
② $E = E_1 < 0$



$r_- < r < r_+$ bounded motion

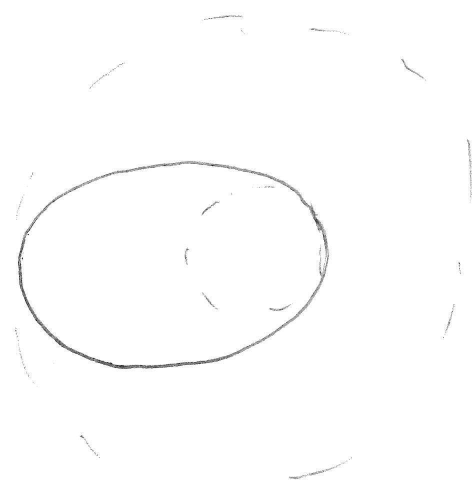
$r_\pm$ - apsidal distances [turning points]

$T_{radial} < T_{angular}$



$T_{radial} > T_{angular}$

~~$T_{angular}$~~



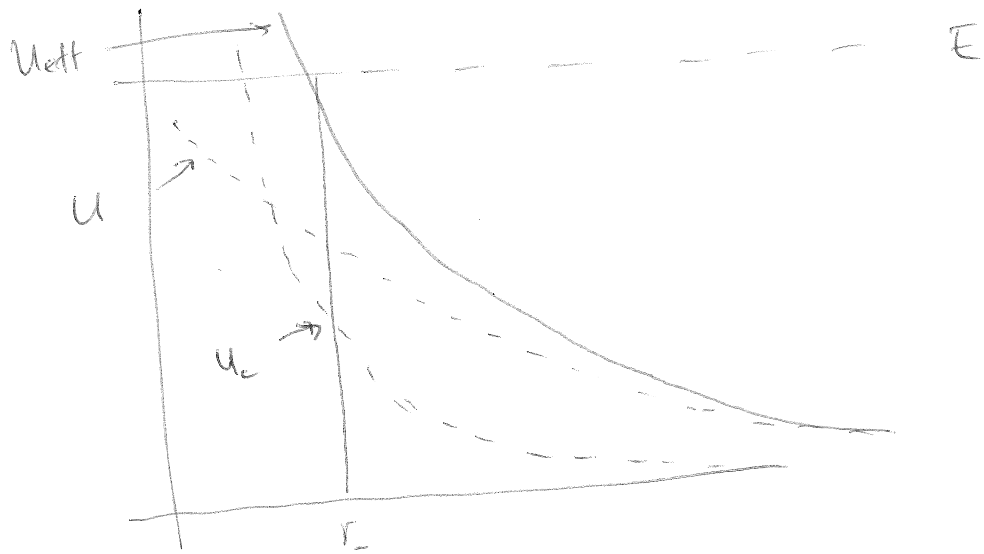
$T_{radial} = T_{angular}$

If T_{radial} and $T_{angular}$ are commensurate, orbit closes.

Bertrand's theorem: the only central forces that result in closed orbits are $f = -\frac{k}{r^2}$ and $f = -kr$ $\left(V = -\frac{k}{r} \text{ and } V = \frac{kr^2}{2} \right)$ ← [w/o proof]

Proof: Goldstein

$U(x)$ repulsive



$E > 0$
$r > r_-$ motion always unbounded

$U(x) = 0$

[looks like repulsion from radial point of view]

[complicated way to view straight line motion]

→ [Quantitative analysis]

$r(t)$, $\theta(t)$: formal solution

$$\frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2mr^2} + U(r) = E$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - U(r) - \frac{\ell^2}{2mr^2}\right)}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{\ell^2}{m^2 r^2}}}$$

$$t(r) = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{\ell^2}{m^2 r^2}}}$$

[analytic or numerical solution]

$$T_{radial} = 2 \int_{r_-}^{r_+} \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{\ell^2}{m^2 r^2}}}$$

Invert to get $r(t)$ [nontrivial]

$$\frac{d\theta}{dt} = \frac{\ell}{m r^2(t)}$$

$$\theta(t) - \theta_0 = \int_0^t \frac{\ell}{m} \frac{dt}{r^2(t)}$$

[everything reduced to quadrature]

Orbit equation: $r(\theta)$

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{\ell}{mr^2}\frac{1}{\sqrt{\frac{2}{m}(E-U(r)) - \frac{\ell^2}{m^2r^2}}}$$

$$\theta(r) - \theta_0 = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2m}{\ell^2}(E-U(r)) - \frac{1}{r^2}}}$$

[non trivial]

Invert to get $r(\theta)$

$$u = \frac{1}{r} \qquad du = -\frac{dr}{r^2}$$

$$\theta(u) - \theta_0 = -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2m}{\ell^2}(E-U) - u^2}}$$

$F \sim r^n \Rightarrow U \sim r^{n+1}$

Simple functions $n = \underset{\uparrow \text{ Hooke}}{1}, \underset{\uparrow \text{ inv sq}}{-2}, -3$

Elliptic ~~int~~ fcns $n = 5, 3, 0, -4, -5, -7$

Inverse square: $V = -\frac{k}{r}$ $k = GMm$ gravity
$= e^2$ hydrogen atom

$$\theta - \theta' = -\int \underbrace{\frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2mk}{\ell^2}u - u^2}}}_{\arccos\left[\frac{\frac{\ell^2 u}{mk} - 1}{\sqrt{1 + \frac{2E\ell^2}{mk^2}}}\right]}$$

$$\frac{1}{r} = u = \frac{mk}{\ell^2}\left[1 + \sqrt{1 + \frac{2E\ell^2}{mk^2}}\,\cos(\theta - \theta')\right]$$

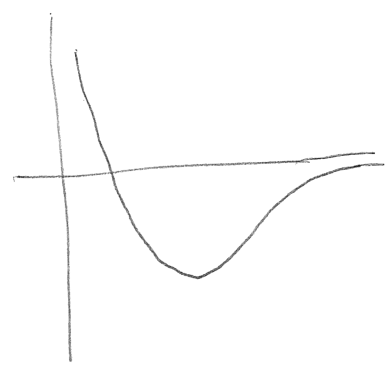
Two parameters (E, ℓ)

$$\frac{1}{r} = C[1 + \underbrace{e}_{\text{eccentricity}} \cos(\theta - \theta')]$$

Conic section

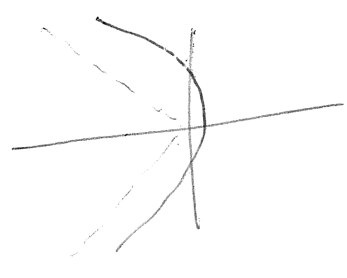
[take $e > 0$, if $e < 0$, shift θ' by π]

$e = 0$	circle	$E = -\frac{mk^2}{2\ell^2}$
$0 < e < 1$	ellipse	$E < 0$
$e = 1$	parabola	$E = 0$
$e > 1$	hyperbola	$E > 0$



Kepler's 1st law

$r \neq 0 \Rightarrow \cos\theta \neq -\frac{1}{e}$



Differential equation for the orbit $r(\theta)$

$$\frac{d\theta}{dt} = \frac{\ell}{mr^2}$$

$$m\frac{d^2r}{dt^2} - \frac{\ell^2}{mr^3} = F(r)$$

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{\ell}{mr^2}\frac{d}{d\theta}$$

$$\begin{cases} r = \dfrac{1}{u} \\ \dfrac{dr}{dt} = -\dfrac{1}{u^2}\dfrac{du}{dt} = -\dfrac{1}{u^2}\dfrac{\ell}{mr^2}\dfrac{du}{d\theta} = -\dfrac{\ell}{m}\dfrac{du}{d\theta} \\ \dfrac{d^2r}{dt^2} = -\dfrac{\ell}{m}\dfrac{d}{dt}\left(\dfrac{du}{d\theta}\right) = -\dfrac{\ell}{m}\dfrac{\ell}{mr^2}\dfrac{d}{d\theta}\left(\dfrac{du}{d\theta}\right) = -\dfrac{\ell^2}{m^2}u^2\dfrac{d^2u}{d\theta^2} \end{cases}$$

$$-\frac{\ell^2}{m}u^2\frac{d^2u}{d\theta^2} - \frac{\ell^2}{m}u^3 = F\left(\frac{1}{u}\right)$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2u^2}F\left(\frac{1}{u}\right)}$$

Given $r(\theta)$, this gives $F(r)$.

$$F = -\frac{k}{r^2} = -ku^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{mk}{l^2}$$

~~Hom~~ Homogeneous: $u = C\cos(\theta - \theta')$

Specific $u = \frac{mk}{l^2}$

General $u = \frac{mk}{l^2}\left[1 + e\cos(\theta - \theta')\right] = \frac{1}{r}$

$\boxed{\text{Ellipse}: 0 < e < 1}$

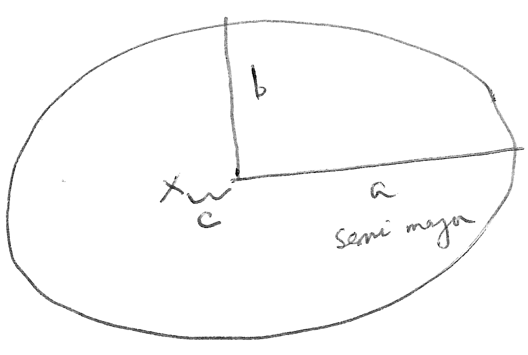
$$\frac{1}{r} = B(1 + e\cos\theta)$$

2 parameters: B, e

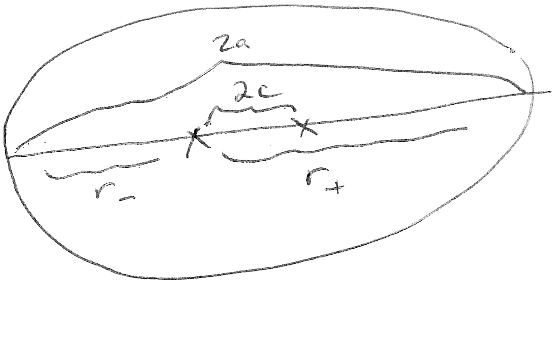
$\theta = 0$ $\quad \frac{1}{r_-} = B(1+e)$

$\theta = \pi$ $\quad \frac{1}{r_+} = B(1-e)$

$$\frac{r_+}{r_-} = \frac{1+e}{1-e} \Rightarrow e = \frac{r_+ - r_-}{r_+ + r_-}$$



$$e \equiv \frac{c}{a} \qquad a^2 = b^2 + c^2$$



$$e = \frac{2c}{2a} = \frac{r_+ - r_-}{r_+ + r_-}$$

Other possible topics

1. Runge Lenz vector

2. Precession of orbit due to GR perturbation
   $\left(\frac{1}{r^4}\right)$

3. Scattering : cross section
   differential cross section
   Rutherford

4. Analysis of scattering in diff. frames, or in lab.

$$B = \frac{1}{r_-(1+e)} = \frac{r_+ + r_-}{2r_+r_-}$$

$$\frac{1}{r} = \frac{r_+ + r_-}{2r_+r_-}\left[1 + \frac{r_+ - r_-}{r_+ + r_-}\cos\theta\right]$$

2 parameters: r_+, r_-

How are r_+, r_- related to E, ℓ?

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{k}{r}$$

$\dot{r} = 0$ at turning points

$$E = \frac{\ell^2}{2mr^2} - \frac{k}{r}$$

$$r^2 + \frac{k}{E}r - \frac{\ell^2}{2mE} = 0$$

$$(r - r_+)(r - r_-) = 0$$

$$r^2 - (r_+ + r_-)r + r_+r_- = 0$$

$$r_\pm = -\frac{k}{2E} \pm \frac{1}{2}\sqrt{\frac{k^2}{E^2} + \frac{2\ell^2}{mE}}$$

$$B = \frac{r_+ + r_-}{2r_+r_-} = \frac{\left(-\frac{k}{E}\right)}{\left(-\frac{\ell^2}{mE}\right)} = \frac{mk}{\ell^2}$$

$$e = \frac{r_+ - r_-}{r_+ + r_-} = \frac{\sqrt{\frac{k^2}{E^2} + \frac{2\ell^2}{mE}}}{-\frac{k}{E}} = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$$

---

semi major axis: $a = \frac{r_+ + r_-}{2} = -\frac{k}{2E}$, $E = -\frac{k}{2a}$ [virial]

$$b = \sqrt{a^2 - c^2} = \sqrt{a^2 - a^2e^2} = a\sqrt{1 - \underbrace{e^2}_{1 + \frac{2E\ell^2}{mk^2}}} = -\frac{k}{2E}\sqrt{-\frac{2E\ell^2}{mk^2}}$$

Area of ellipse $= \pi ab = \frac{\pi k^2}{4E^2}\sqrt{-\frac{2E\ell^2}{mk^2}}$

Period of motion

$$\frac{dA}{dt} = \frac{l}{2m}$$

$$A = \frac{l}{2m}\tau$$

$$\tau = \frac{2mA}{l} = \pi k \sqrt{\frac{m}{2}}\sqrt{\frac{1}{-E^3}}$$

$$\tau^2 = \frac{\pi^2 k^2 m}{-2E^3} = \frac{4\pi^2 m}{k} a^3$$

$$\boxed{\tau^2 = \frac{4\pi^2 m}{k} a^3}$$ Kepler's 3rd law

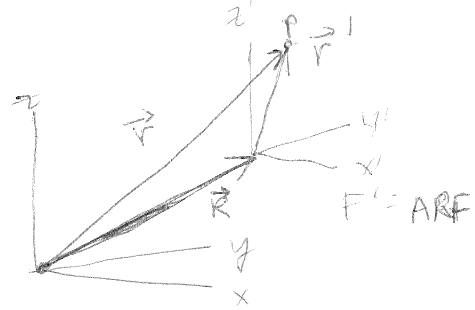
Velocities

$$E = \frac{1}{2}mv^2 - \frac{k}{r} = -\frac{k}{(r_+ + r_-)}$$

At perihelion, aphelion

$$\frac{1}{2}mv_\pm^2 = \frac{k}{r_\pm} - \frac{k}{(r_+ + r_-)}$$

Non inertial (accelerating) reference frames



F = IRF

$$\vec{r}' = \vec{r} - \vec{R}$$

$$\vec{v}' = \vec{v} - \vec{v}_R$$

$$\vec{a}' = \vec{a} - \vec{a}_R$$

F : $\quad m\vec{a} = \vec{F}$

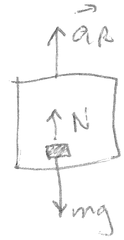
$$m(\vec{a}' + \vec{a}_R) = \vec{F}$$

$$m\vec{a}' = \vec{F} - m\vec{a}_R$$

$$= \vec{F} + \vec{F}_{pseudo}$$

$$\vec{F}_{pseudo} = -m\vec{a}_R$$

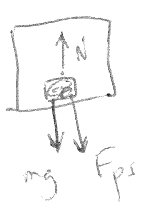
elevator



$$N - mg = ma_R$$

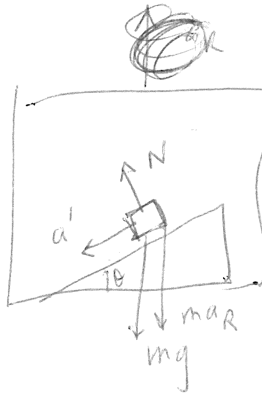
$$N = m(g + a_R)$$

ARF



$$N - mg - ma_R = 0$$

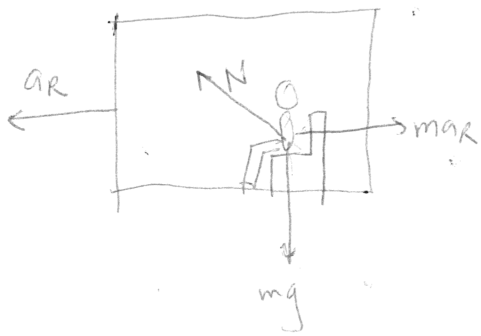
$$N = m(g + a_R)$$



ARF

$$N = m(g + a_R)\cos\theta$$

$$m(g + a_R)\sin\theta = ma'$$



[which way does helium balloon go?]