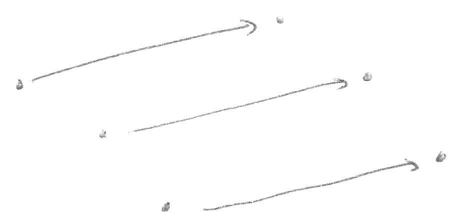
① KI

Conservation laws and symmetry

① Homogeneity: Physics is invariant under translation of entire system

$$\vec{r}_i \longrightarrow \vec{r}_i + \vec{R}$$

$$\dot{\vec{r}}_i \longrightarrow \dot{\vec{r}}_i$$



$$L(\vec{r}_i, \dot{\vec{r}}_i) = L(\vec{r}_i + \vec{R}, \dot{\vec{r}}_i)$$

Let $\vec{R}$ be infinitesimal $L(\vec{r}_i, \dot{\vec{r}}_i) = L(\vec{r}_i + d\vec{R}, \dot{\vec{r}}_i)$

$$= \underbrace{L(\vec{r}_i, \dot{\vec{r}}_i) + \sum_i \frac{\partial L}{\partial \vec{r}_i} \cdot \underbrace{d\vec{r}_i}_{d\vec{R}}}_{d\vec{R}\cdot\left(\sum \frac{\partial L}{\partial \vec{r}_i}\right)} + \dots$$

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = 0$$

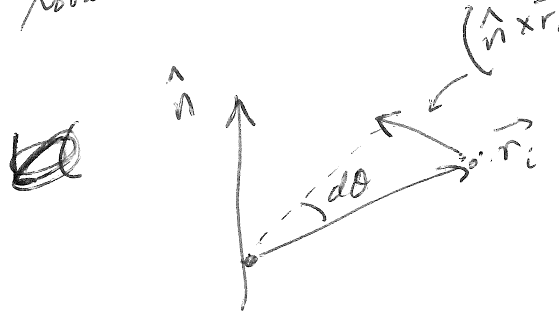
Euler Lagr. $\quad \frac{d}{dt} \sum \left(\frac{\partial L}{\partial \dot{\vec{r}}_i} \right) = \frac{d}{dt} \sum (\underbrace{m\dot{\vec{r}}_i}_{\vec{p}_i}) = \frac{d}{dt} \vec{P} = 0$

$L = \frac{1}{2} \sum m \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$

$\frac{\partial L}{\partial \dot{\vec{r}}_i} = m\dot{\vec{r}}_i$

$$\boxed{\text{Translation invariance} \Rightarrow \vec{P} \text{ conserved}}$$

② Isotropy: physics is invariant under a rotation of the entire system



$$L(\vec r_i) = L(\vec r_i + \hat n \times \vec r_i\, d\theta,\ \dot{\vec r}_i + \hat n \times \dot{\vec r}_i\, d\theta)$$

$$= L(\vec r_i) + \sum \underbrace{\frac{\partial L}{\partial \vec r_i}\cdot(\hat n \times \vec r_i)\,d\theta}_{\sum \hat n \cdot \left(\vec r_i \times \frac{\partial L}{\partial \vec r_i}\right) d\theta + \dots} + \sum \frac{\partial L}{\partial \dot{\vec r}_i}\cdot(\hat n \times \dot{\vec r}_i) + \dots$$

$$\hat n\, d\theta \cdot \sum\left(\vec r_i \times \frac{\partial L}{\partial \vec r_i}\right) + \dot{\vec r}_i \times \frac{\partial L}{\partial \dot{\vec r}_i}$$

$$0 = \sum_i \vec r_i \times \frac{\partial L}{\partial \vec r_i} + \dots = \sum\left(\vec r_i \times \frac{d\vec p_i}{dt} + \frac{dr_i}{dt} \times \vec p_i\right)$$

$$= \frac{d}{dt}\left(\sum \vec r_i \times p_i\right)$$

$$= \frac{d}{dt}\vec L$$

$$\vec L = \text{const}$$

Rotation invariance $\Rightarrow$ conservation of angular momentum

[conservation of energy? think relativity!]

③ Homogeneity in time: physics is invariant under translation in time $\Rightarrow$ L not an explicit function of time

$$\frac{\partial L}{\partial t} = 0$$

[can't say $\frac{dL}{dt} = 0$ because $q = q(t)$]

$$\frac{dL}{dt} = \underbrace{\underbrace{\frac{\partial L}{\partial q_i}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \underbrace{\ddot{q}_i}_{\frac{d}{dt}(\dot{q}_i)}}_{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right]} + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right] + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left[\underbrace{\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L}_{\equiv h}\right] = -\frac{\partial L}{\partial t}$$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dh}{dt} = 0 \Rightarrow h = \text{conserved}$$

Energy

time translation invariance $\Rightarrow$ energy conservation

h conserved if $\frac{\partial L}{\partial t} = 0$,

Does $h = T + V$?

Yes, if (a) V is velocity independent [not true if magnetic field]

(b) constraints are scleronomic,

ie relations between Cartesian + generalized coords do <u>not</u> depend on time

$x_i = x_i(q, \cancel{t})$

(a) $\Rightarrow \frac{\partial V}{\partial \dot{q}_i} = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$

(b) $\Rightarrow T = a_{jk}(q)\, \dot{q}_j \dot{q}_k + \underbrace{\cancel{b_i \dot{q}_i}}_{0} + \underbrace{\cancel{c}}_{0}$ since $\frac{\partial x_i}{\partial t} = 0$

$$\frac{\partial T}{\partial \dot{q}_i} = a_{jk} \dot{q}_j \underbrace{\frac{\partial \dot{q}_k}{\partial \dot{q}_i}}_{\delta_{ki}} + a_{jk} \underbrace{\frac{\partial \dot{q}_j}{\partial \dot{q}_i}}_{\delta_{ji}} \dot{q}_k$$

$$= a_{ji} \dot{q}_j + a_{ik} \dot{q}_k$$

$$= 2 a_{ij} \dot{q}_j$$

$$\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2 a_{ij} \dot{q}_i \dot{q}_j = 2T$$

$$h = 2T - L = T + V \quad ✓$$

$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$ = ~~moment~~ (generalized) momentum

~~conjugate to q_i~~

= canonical momentum

conjugate momentum

momentum conjugate to q_i

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$$

~~If $\frac{\partial L}{\partial q_i} = 0$ then q_i is cyclic~~

q_i is a cyclic coordinate if $\frac{\partial L}{\partial q_i} = 0$

an ignorable

The momentum conjugate to a cyclic coordinate is conserved

Hamiltonian: $H = \left(\sum_i p_i \dot{q}_i\right) - L$

Numerically the same as h = energy...

H appears to depend on $q_i, \dot{q}_i, p_i$

$$dH = dp_i\, \dot{q}_i + \cancel{p_i\, d\dot{q}_i} - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} dq_i - \cancel{\underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

$$dH = dp_i\, \dot{q}_i - \dot{p}_i\, dq_i - \frac{\partial L}{\partial t} dt$$

H only depends on p_i, q_i, and t

Hamiltons eqn's (2 1st order)

$$\frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i$$

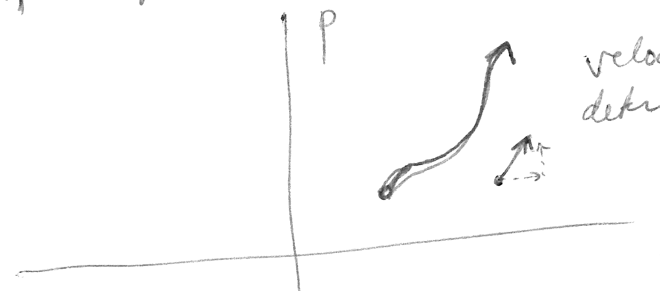
$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\frac{dH}{dt} = \underbrace{\underbrace{\frac{\partial H}{\partial p_i}}_{+\dot{q}_i} \dot{p}_i + \underbrace{\frac{\partial H}{\partial q_i}}_{-\dot{p}_i} \dot{q}_i}_{0} + \underbrace{\frac{\partial H}{\partial t}}_{-\frac{\partial L}{\partial t}} = -\frac{\partial L}{\partial t}$$

$\frac{\partial L}{\partial t} = 0 \Rightarrow H$ const.

<u>Phase space</u> : 2n dimensional space describing state of the system

eg <u>n=1</u> (p,q) space



velocity in phase space determined by position

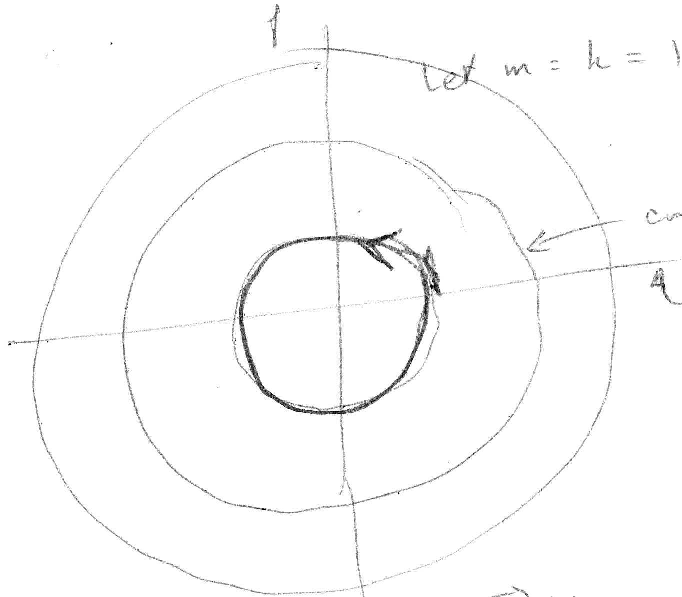
$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

eg harmonic oscillator : $H = T + V = \frac{p^2}{2m} + \frac{1}{2} k q^2$

$H(p,q)$

let $m = k = 1$, $H = \frac{1}{2}(p^2+q^2)$



constant energy surfaces

gradient of H = $\left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) = \vec{\nabla} H$

direction of flow = $(\dot{q}, \dot{p}) = \dot{\vec{X}}$

Consider $\dot{\vec{X}} \cdot \vec{\nabla} H = \dot{q}\frac{\partial H}{\partial q} + \dot{p}\frac{\partial H}{\partial p} = -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} = 0$

$\perp$ $\Rightarrow$ flow is $\perp$ to gradient

ie along lines of const H

virial thm (statistical)

Consider a collection of particles $\vec{r}_i$ and $\vec{p}_i$ confined to some finite region of phase space. (ie they ~~stick~~ stay together)

Define $S = \sum_i \vec{p}_i \cdot \vec{r}_i \quad \Rightarrow \quad S$ is bounded

$$\frac{dS}{dt} = \sum \left(p_i \cdot \dot{\vec{r}}_i + \dot{\vec{p}}_i \cdot \vec{r}_i \right)$$

Time average

$$\left\langle \frac{dS}{dt} \right\rangle = \frac{1}{T} \int_0^T \frac{dS}{dt}\, dt = \frac{S(T) - S(0)}{T}$$

$$\left| S(T) - S(0) \right| < \text{const}$$

$$\left| \frac{S(T) - S(0)}{T} \right| \xrightarrow[T \to \infty]{} 0$$

$$\left\langle \frac{dS}{dt} \right\rangle \xrightarrow[T \to 0]{} 0$$

$$\underbrace{\left\langle \sum \underbrace{p_i}_{m\dot{\vec{r}}_i} \cdot \dot{\vec{r}}_i \right\rangle}_{\langle 2T \rangle} = - \left\langle \underbrace{\sum \dot{\vec{p}}_i \cdot \vec{r}_i}_{\sum \vec{F}_i \cdot \vec{r}_i} \right\rangle \quad (T \to \infty)$$

$$\langle T \rangle = \underbrace{-\frac{1}{2} \left\langle \sum \vec{F}_i \cdot \vec{r}_i \right\rangle}_{\text{virial of the system}}$$

Let particles interact via a central force $\vec{F} = -k r^n \hat{r}$

then $\vec{F} = -\vec{\nabla} U$, $U = \frac{k}{n+1} r^{n+1}$

$$U_{int} = \sum_{pairs} \frac{k}{n+1} r_{ij}^{n+1}$$

$$\sum_i \vec{r}_i \cdot \vec{F}_i = \sum_{\substack{i \\ j \neq i}} \vec{F}_{ij} \cdot \vec{r}_i = \sum_{i<j} \vec{F}_{ij} \cdot \vec{r}_i + \underbrace{\sum_{i>j} \underbrace{\vec{F}_{ij}}_{-\vec{F}_{ji}} \cdot \vec{r}_j}_{-\sum_{i<j} \vec{F}_{ij} \cdot \vec{r}_i} = \sum_{i<j} \vec{F}_{ij} \cdot \underbrace{(\vec{r}_i - \vec{r}_j)}_{r_{ij} \hat{r}_{ij}}$$

$$= \sum_{i<j} \left[-k \, r_{ij}^{n} \hat{r}_{ij} \right] \cdot r_{ij} \hat{r}_{ij}$$

$$= \sum_{pairs} -k \, r_{ij}^{n+1} = -(n+1) U$$

$$\langle T \rangle = -\frac{1}{2} \langle \sum \vec{F}_i \cdot \vec{r} \rangle = \frac{n+1}{2} \langle U \rangle$$

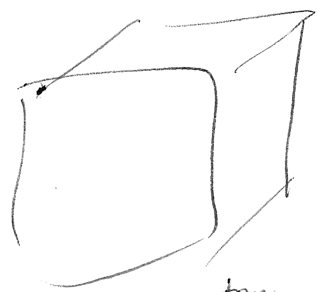
gravity: $n = -2 \Rightarrow \langle T \rangle = -\frac{1}{2} \langle U \rangle$

$$\frac{mv^2}{r} = + \frac{GMm}{r^2}$$

$$\frac{1}{2} m v^2 = + \frac{1}{2} \frac{GMm}{r} = -\frac{1}{2} \left(-\frac{GMm}{r} \right) \checkmark$$

harm osc: $n = 1 \Rightarrow \langle T \rangle = \langle U \rangle$

Derivation of ideal gas law



ideal, monatomic

what are forces?

force of constraint.

collision forces cancel out ~~because particles~~

$\vec{F}_{ij} = -\vec{F}_{ji}$

because pcls are at same place $\vec{r}_i = \vec{r}_j$

i.e. $\underbrace{(\vec{r}_i - \vec{r}_j)}_{0 \text{ when } \vec{F}_{ij} \neq 0} \cdot \vec{F}_{ij} = 0$

$$\underbrace{\langle T \rangle}_{\frac{3}{2} NkT} = -\frac{1}{2} \langle \sum \vec{F}_i \cdot \vec{r}_i \rangle$$

$\underbrace{\sum \vec{F}_i}_{-P\, d\vec{A}}$ since $d\vec{A} \sim \hat{n}$

Force imparted by wall per unit area $= P$

$P\, dA\, dt = \sum F_i\, dt$

$$= \frac{1}{2} P \underbrace{\int d\vec{A} \cdot \vec{r}}_{\int \underbrace{\vec{\nabla} \cdot \vec{r}}_{3}\, dV}$$

$= \frac{3}{2} PV$

$PV = NkT$

[what abt diatomic $\frac{5}{2} kT$, $\frac{7}{2} kT$]

(combining T & U)

Rotational coordinates not bounded since no restoring force ...