Lagrangian formulation of mechanics

- Specify configuration of system using generalized coordinates

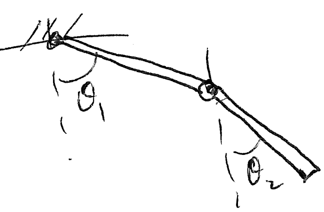
$q_i \qquad i = 1, \dots, n$

$\dot{q}_i$ = generalized velocities

Ex

1) Point particle $\vec{r} = (x, y, z)$
$= (r, \theta, \phi)$
$= (\rho, \phi, z)$

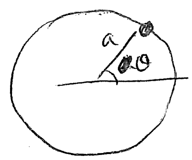
2) Double pendulum θ_1, θ_2



- Constraints ~~[illegible]~~

(I) holonomic: equalities among the coordinates only

e.g. particle constrained to move on a circle



~~constraint~~

$x^2 + y^2 = a^2$

→ may be eliminated by choosing a smaller set of independent generalized coordinates

e.g. θ

→ or use Lagrange multipliers

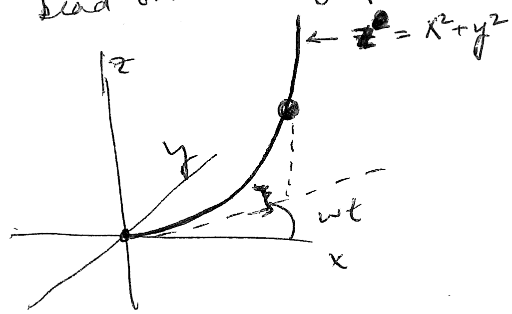
(2) nonholonomic: equalities ~~among~~ involving velocities, or inequalities

eg particle moving on top of a solid hemisphere

$x^2+y^2+z^2 \geq a^2$

- constraints may involve time (rheonomic)

eg. bead on rotating parabolic wire



$x = \xi \cos \omega t$
$y = \xi \sin \omega t$
$z = \xi^2$

if not, scleronomic

- Define Lagrangian function $L = T - U$

- Express T and U in terms of $q_i, \dot{q}_i$

<u>Point particle</u>

$$T = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 = \frac{1}{2} m \dot{x}_i \dot{x}_i$$

Einstein summation convention: repeated indices are summed over

$$x_i = x_i(q, t) \equiv x_i(q_1, q_2, \ldots, q_n, t)$$

$$\frac{dx_i}{dt} = \frac{\partial x_i}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial x_i}{\partial t}$$

$$T = \frac{1}{2} m \left(\frac{\partial x_i}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial x_i}{\partial t} \right) \left(\frac{\partial x_i}{\partial q_k} \frac{dq_k}{dt} + \frac{\partial x_i}{\partial t} \right)$$

$$= \underbrace{\frac{1}{2} m \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_k}}_{a_{jk}} \dot{q}_j \dot{q}_k + \underbrace{m \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial t}}_{b_j} \dot{q}_j + \underbrace{\frac{1}{2} m \frac{\partial x_i}{\partial t} \frac{\partial x_i}{\partial t}}_{c}$$

$$T(q, \dot{q}, t) = a_{jk}(q, t)\, \dot{q}_j \dot{q}_k + b_j(q, t)\, \dot{q}_j + c(q, t)$$

<u>Ex</u> Bead on wire

$$x = \xi \cos\omega t$$
$$y = \xi \sin\omega t$$
$$z = \xi^2$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\Big[(\dot{\xi}\cos\omega t - \xi\,\omega\sin\omega t)^2 + (\dot{\xi}\sin\omega t - \xi\,\omega\cos\omega t)^2 + (2\xi\dot{\xi})^2\Big]$$

$$= \frac{1}{2}m\left[\dot{\xi}^2 - \omega^2\xi^2 + 4\xi^2\dot{\xi}^2\right]$$

$$= \underbrace{\frac{1}{2}m(1 + 4\xi^2)}_{a(\xi)}\dot{\xi}^2 - \underbrace{\frac{1}{2}m\omega^2\xi^2}_{c(\xi)}$$

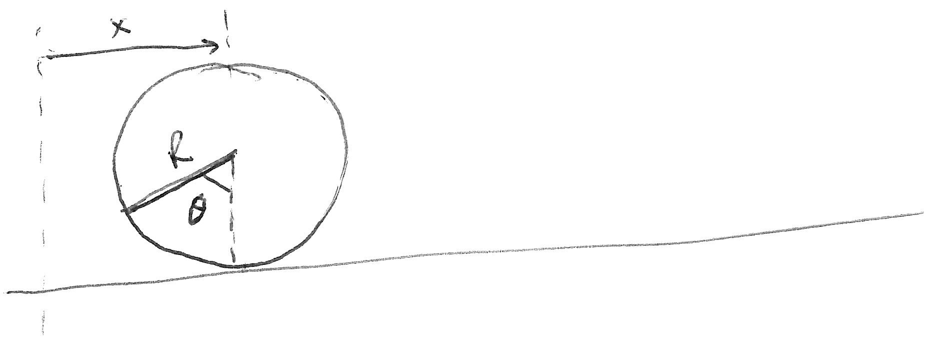
If generalized coordinates do not depend <u>explicitly</u> on time, then T is quadratic in the generalized velocities; otherwise linear + constant terms occur as well.

$$T = T(q, \dot{q}, t)$$
$$U = U(q, \dot{q}, t)$$

$$\Rightarrow L = L(q, \dot{q}, t)$$

Constraint:

rolling w/o slipping



$$T = T_{cm} + T_{\text{w.r.t. cm}}$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

Holonomic constraint : $x = R\theta$

$$\dot{x} = R\dot{\theta}$$

$$= \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$= \frac{1}{2} (I + mR^2) \dot{\theta}^2$$

Extensive discussion of

$$T = a_{ij} \dot{q}_i \dot{q}_j$$

$$\Downarrow$$

$$(\dot{q}_1 \cdots \dot{q}_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

[students not familiar w/ matrix notat to write sums]

Or even dot product $\vec{v} \cdot \vec{w} = (v_1 \cdots v_n) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

ex

particle in polar coordinates

$$x = r\cos\theta \qquad \frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$y = r\sin\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$q_1 = r \qquad x_1 = x$$
$$q_2 = \theta \qquad x_2 = y$$

$$a_{jk} = \frac{m}{2}\left(\frac{\partial x_i}{\partial q_j}\frac{\partial x_i}{\partial q_k}\right)$$

$$a_{11} = \frac{m}{2}\left(\frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r}\right) = \frac{m}{2}(\cos^2\theta + \sin^2\theta) = \frac{m}{2}$$

$$a_{12} = \frac{m}{2}\left(\frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta}\right) = \frac{m}{2}\left(-r\sin\theta\cos\theta + r\sin\theta\cos\theta\right) = 0$$

$$a_{21} = a_{12} = 0$$

$$a_{22} = \frac{m}{2}\left(\frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta}\right) = \frac{m}{2}(r^2\sin^2\theta + r^2\cos^2\theta) = \frac{m}{2}r^2$$

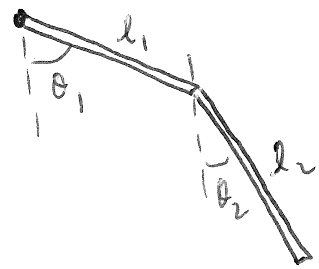
$$T = a_{11}\dot{q}_1\dot{q}_1 + a_{22}\dot{q}_2\dot{q}_2 \,, \qquad b_i = c = 0 \text{ bc } \frac{\partial x_i}{\partial t} = 0$$

$$= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

Derivation
Lorentz force from Lagrangian

I am not sure if students will do too well since likely they have not heard of vector potential omit

Double pendulum



Center of mass

$$\begin{cases} x_1 = \frac{l_1}{2}\sin\theta_1 \\ y_1 = -\frac{l_1}{2}\cos\theta_1 \end{cases}$$

$$\begin{cases} x_2 = l_1\sin\theta_1 + \frac{l_2}{2}\sin\theta_2 \\ y_2 = -l_1\cos\theta_1 - \frac{l_2}{2}\cos\theta_2 \end{cases}$$

$$T = \frac{1}{2}m_1\left(\frac{l_1^2}{4}\right)\dot\theta_1^2 + \frac{1}{2}\left(\frac{1}{12}m_1 l_1^2\right)\dot\theta_1^2$$

$$+ \frac{1}{2}m_2\Big[\underbrace{\left(l_1\cos\theta_1\dot\theta_1 + \frac{l_2}{2}\cos\theta_2\dot\theta_2\right)^2 + \left(l_1\sin\theta_1\dot\theta_1 + \frac{l_2}{2}\sin\theta_2\dot\theta_2\right)^2}_{l_1^2\dot\theta_1^2 + \frac{l_2^2}{4}\dot\theta_2^2 + l_1 l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2}\Big] + \frac{1}{2}\left(\frac{1}{12}m_2 l_2^2\right)\dot\theta_2^2$$

$$= \frac{1}{2}\left(m_2 + \frac{1}{3}m_1\right)l_1^2\dot\theta_1^2 + \frac{1}{6}m_2 l_2^2\dot\theta_2^2 + \frac{1}{2}m_2 l_1 l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2$$

$$U = -m_1 g\frac{l_1}{2}\cos\theta_1 - m_2 g\left(l_1\cos\theta_1 + \frac{l_2}{2}\cos\theta_2\right)$$

$$= -g l_1\left(m_2 + \frac{m_1}{2}\right)\cos\theta_1 - g l_2\frac{m_2}{2}\cos\theta_2$$

$$\frac{\partial L}{\partial \dot\theta_1} = \left(m_2 + \tfrac{1}{3}m_1\right) l_1^2 \dot\theta_1 + \frac{m_2}{2} l_1 l_2 \cos(\theta_1 - \theta_2)\, \dot\theta_2$$

$$\frac{\partial L}{\partial \dot\theta_2} = \frac{m_2}{3} l_2^2 \dot\theta_2 + \frac{m_2}{2} l_1 l_2 \cos(\theta_1 - \theta_2)\, \dot\theta_1$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\, \dot\theta_1 \dot\theta_2 - g l_1 \left(m_2 + \frac{m_1}{2}\right) \sin\theta_1$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2} m_2 l_1 l_2 \sin(\theta_2 - \theta_1)\, \dot\theta_1 \dot\theta_2 - g l_2 \left(\frac{m_2}{2}\right) \sin\theta_2$$

$$\left(m_2 + \frac{m_1}{3}\right) l_1^2 \ddot\theta_1 + \frac{m_2}{2} l_1 l_2 \cos(\theta_1 - \theta_2)\, \ddot\theta_2 - \underbrace{\frac{m_2}{2} l_1 l_2 \sin(\theta_1 - \theta_2)\, \dot\theta_1 \dot\theta_2}$$

$$= -\underbrace{\frac{1}{2} m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\, \dot\theta_1 \dot\theta_2} - g l_1 \left(m_2 + \frac{m_1}{2}\right) \sin\theta_1$$

$$\boxed{\left(m_2 + \frac{m_1}{3}\right) l_1 \ddot\theta_1 + \frac{m_2}{2} l_2 \cos(\theta_1 - \theta_2)\, \ddot\theta_2 + g l_1 \left(m_2 + \frac{m_1}{2}\right) \sin\theta_1 = 0}$$

$$\frac{m_2}{3} l_2^2 \ddot\theta_2 + \frac{m_2}{2} l_1 l_2 \cos(\theta_1 - \theta_2)\, \ddot\theta_1 - \underbrace{\frac{m_2}{2} l_1 l_2 \sin(\theta_2 - \theta_1)\, \dot\theta_2 \dot\theta_1}$$

$$= -\underbrace{\frac{1}{2} m_2 l_1 l_2 \sin(\theta_2 - \theta_1)}\, \dot\theta_1 \dot\theta_2 - g l_2 \frac{m_2}{2} \sin\theta_2$$

$$\boxed{\frac{m_2}{3} l_2 \ddot\theta_2 + \frac{m_2}{2} l_1 \cos(\theta_1 - \theta_2)\, \ddot\theta_1 + g l_2 \frac{m_2}{2} \sin\theta_2 = 0}$$

[Since this was all new to them, I decided to re-emphasize it;]

$x_i(q_j)$

Assume same no d.o.f. & non-time dep.

$$\dot{x}_i = \frac{\partial x_i}{\partial q_j}\dot{q}_j$$

$x = r\cos$
$y = r\sin$

$$\Lambda_{ij} = \frac{\partial x_i}{\partial q_j}$$

$$\dot{x}_i = \Lambda_{ij}\dot{q}_j$$

$$\Lambda = \begin{pmatrix} \cos & -r\sin \\ \sin & r\cos \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \vdots \end{pmatrix}$$

$V_i^x = \Lambda_{ij} V_j^q$ contravariant vector

$$T = \frac{1}{2}m\,\dot{x}_i\dot{x}_i = \frac{1}{2}m(\dot{x}_1 \cdots)\begin{pmatrix} \dot{x}_1 \\ \vdots \end{pmatrix} = \frac{1}{2}m(\dot{x}_1 \cdots)\underbrace{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}_{A_x}\begin{pmatrix} \dot{x}_1 \\ \vdots \end{pmatrix}$$

$$= \frac{1}{2}m\,\dot{x}A_x\dot{x}$$

$$= \frac{1}{2}m\,\dot{q}^T\underbrace{\Lambda^T A_x \Lambda}_{A_q}\dot{q}$$

$(AB)^T = B^T A^T$

$\dot{x}^T = \dot{q}^T\Lambda^T$

$$A_q = \Lambda^T A_x \Lambda$$

$$= \begin{pmatrix} \cos & \sin \\ -r\sin & r\cos \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & -s \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$$

$$T = \frac{1}{2}m(\dot{r}\ \dot{\theta})\begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

- Define action $S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)$

~~[illegible]~~

Hamilton's principle: a system follows path of least action,

ie $q(t)$ minimizes $S[q(t)]$

~~[illegible]~~
$$
\begin{aligned}
x &\to t \\
y &\to q \\
y' &\to \dot{q} \\
f &\to L \\
S &\to J
\end{aligned}
$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \dots n$$

Euler-Lagrange eqns

Ex

Bead on wire

$$L = \frac{1}{2}m(1+4\zeta^2)\dot{\zeta}^2 + \frac{1}{2}m\omega^2\zeta^2 - mg\zeta^2$$

$$\frac{\partial L}{\partial \dot{\zeta}} = m(1+4\zeta^2)\dot{\zeta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\zeta}}\right) = m(1+4\zeta^2)\ddot{\zeta} + 8m\zeta\dot{\zeta}^2$$

$$\frac{\partial L}{\partial \zeta} = 4m\zeta\dot{\zeta}^2 + m\omega^2\zeta - 2mg\zeta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\zeta}}\right) - \frac{\partial L}{\partial \zeta} = m\left[(1+4\zeta^2)\ddot{\zeta} + 4\zeta\dot{\zeta}^2 - \omega^2\zeta + 2g\zeta\right] = 0$$

[easier to calc $T + V$, than $\vec{F}$!]

Ex

Free Particle in ~~cylindrical~~ cylindrical coordinates

$$\dot{\vec{r}} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} +$$

$$v^2 = \dot{\rho}^2 + (\rho\dot{\phi})^2 +$$

$$L = T - V = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 +) - V(\rho, \phi,)$$

$$\vec{\nabla}V = \left(\frac{\partial V}{\partial \rho}, \frac{1}{\rho}\frac{\partial V}{\partial \phi}, \right) = -\vec{F}$$

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{\rho}} &= m\dot{\rho} \\ \frac{\partial L}{\partial \rho} &= m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho} \end{aligned}\right\} \quad \begin{aligned} & m\ddot{\rho} - m\rho\dot{\phi}^2 + \frac{\partial V}{\partial \rho} = 0 \\ & m(\ddot{\rho} - \rho\dot{\phi}^2) = -\frac{\partial V}{\partial \rho} = F_\rho \end{aligned}$$

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{\phi}} &= m\rho^2\dot{\phi} \\ \frac{\partial L}{\partial \phi} &= -\frac{\partial V}{\partial \phi} \end{aligned}\right\} \quad \begin{aligned} & m(\rho^2\ddot{\phi} + 2\rho\dot{\rho}\dot{\phi}) + \frac{\partial V}{\partial \phi} = 0 \\ & m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) = -\frac{1}{\rho}\frac{\partial V}{\partial \phi} = F_\phi \end{aligned}$$

$$\cancel{\left.\begin{aligned} \frac{\partial L}{\partial \dot{z}} &= m\dot{z} \\ \frac{\partial L}{\partial z} &= -\frac{\partial V}{\partial z} \end{aligned}\right\} \quad \begin{aligned} & m\ddot{z} + \frac{\partial V}{\partial z} = 0 \\ & m\ddot{z} = -\frac{\partial V}{\partial z} = F_z \end{aligned}}$$

$$\cancel{\vec{\nabla}V = \left(\frac{\partial V}{\partial \rho}, \frac{1}{\rho}\frac{\partial V}{\partial \phi}, \frac{\partial V}{\partial z}\right) = -\vec{F}}$$

[much quicker than ~~actually~~ ~~Newton's law~~ deriving cyl components of acceleration]

Advantages of Lagrangian formulation

~~1~~ 1.) Euler Lagrange eqns valid for any system of coordinates
(eg spherical)

2) easier to calculate T + V than components of force

3) Holonomic constraints easy to choose

↳ ignore constraints

~~3)~~ ~~Constraint~~ ...

In ~~Newtonian~~ : ne

Newtonian: need to know forces that enforce constraint..

Frivolous discussion of

S^2 $x^2+y^2+z^2=R^2$
S^1 $x^2+y^2=R^2$
S^0 $x^2=R^2 \Rightarrow x=\pm R$

$S^3 = x^2+y^2+\cancel{z}^2+t^2=R^2$

How to see S^3 by scanning in time

Jim Dugan: Why do we live in 3 dimensions?

→ discussion of string theory 26, 10 dimensions
compactification
How to see compactified dimensions

(~ proton decay)

Lagrange multipliers: when need the force of constraint

~~Lagrange multipliers~~

~~useful for finding force of constraint~~

coordinates q_i, $i = 1, \ldots, n$

Consider the variation $q_i(\alpha) = \bar{q}_i + \alpha \eta_i$ for arbitrary η_i

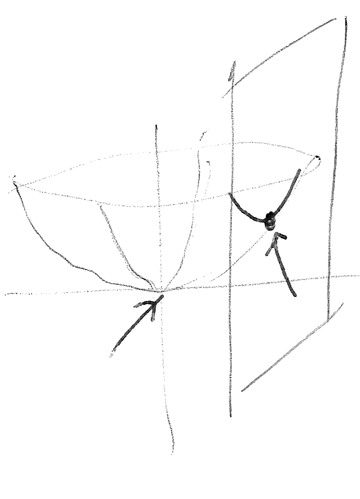
$$S[q_i(\alpha)] \geq S[\bar{q}_i]$$

$$\frac{dS}{d\alpha} = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \eta_i = 0$$

If η_i all <u>independent</u>, then $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$, $i = 1 \ldots n$

Constraints

Suppose q_i are <u>not</u> independent, but obey $f(q_1, q_2 \ldots) = 0$



Consider only variations consistent w/ constraint:

$$0 = f(q_1(\alpha), q_2(\alpha) \ldots)$$

$$0 = \frac{df}{d\alpha}(q_1(\alpha), \ldots) = \frac{\partial f}{\partial q_i} \frac{\partial(\bar{q}_i + \alpha \eta_i)}{\partial \alpha}$$

$$= \frac{\partial f}{\partial q_i} \eta_i$$

$\Rightarrow \eta_i$ <u>not</u> independent...

$$0 = \frac{\partial f}{\partial q_i}\eta_i = \left(\sum_{i=1}^{n-1} \frac{\partial f}{\partial q_i}\eta_i\right) + \frac{\partial f}{\partial q_n}\eta_n$$

$$\eta_n = \frac{-\sum_1^{n-1}\frac{\partial f}{\partial q_i}\eta_i}{\frac{\partial f}{\partial q_n}}$$

$\eta_1, \ldots, \eta_{n-1}$ independent; η_n determined

$$\cancel{\frac{dS}{d\alpha} = \int_{t_1}^{t_2} dt \left[\sum_1^{n-1}\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\right)\eta_i\right] + \left(\frac{\partial L}{\partial q_n} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_n}\right)\right)}$$

Let $D_i = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)$

$$\frac{dS}{d\alpha} = \int dt \sum_{i=1}^{n-1} D_i\eta_i + \underbrace{D_n\eta_n}_{-\sum\frac{\partial f}{\partial q_i}\eta_i \frac{D_n}{\left(\frac{\partial f}{\partial q_n}\right)}}$$

$$= \int dt \sum_{i=1}^{n-1}\left[D_i - \frac{\left(\frac{\partial f}{\partial q_i}\right)}{\left(\frac{\partial f}{\partial q_n}\right)} D_n\right]\eta_i$$

η_i, $i = 1, \ldots n-1$ independent

$$D_i - \frac{\left(\frac{\partial f}{\partial q_i}\right)}{\left(\frac{\partial f}{\partial q_n}\right)} D_n = 0$$

$$\frac{D_i}{\left(\frac{\partial f}{\partial q_i}\right)} - \underbrace{\frac{D_n}{\left(\frac{\partial f}{\partial q_n}\right)}} = 0 \qquad i = 1, \ldots n-1$$

$$\cancel{D_i = -\lambda \frac{\partial f}{\partial q_i}}$$

Lagrange multiplier

Define $\lambda(t) = -\dfrac{D_n}{\left(\frac{\partial f}{\partial q_n}\right)} \Rightarrow D_n + \lambda \dfrac{\partial f}{\partial q_n} = 0$

$$\frac{D_i}{\left(\frac{\partial f}{\partial q_i}\right)} + \lambda(t) = 0 \quad \Rightarrow \quad D_i + \lambda \frac{\partial f}{\partial q_i} = 0$$

$i = 1 .. n-1$

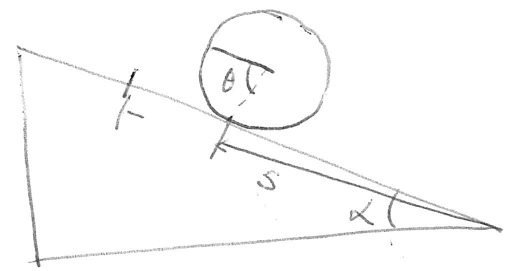
$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \lambda \frac{\partial f}{\partial q_i} = 0$$

$L' = L + \lambda f$

$$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) = 0$$

$\Rightarrow$ constraints $f(q) = 0$ can be incorporated by adding $\lambda(t) f(q)$ to L

$\lambda \dfrac{\partial f}{\partial q_i}$ represents the force of constraint on the coordinate q_i



$$L = \tfrac{1}{2} m \dot{s}^2 + \tfrac{1}{2} I \dot{\theta}^2 - mg\, s \sin\alpha$$

Either:

① eliminate constraint:

$$R\dot{\theta} = \dot{s}$$

$$L = \tfrac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{s}^2 - mg\, s \sin\alpha$$

$$\Rightarrow \left(m + \frac{I}{R^2}\right)\ddot{s} = -mg\sin\alpha$$

② lagrange multiplier: $f = R\theta - s = 0$

$$L' = L + \lambda(R\theta - s)$$

$$= \tfrac{1}{2} m\dot{s}^2 + \tfrac{1}{2} I\dot{\theta}^2 - mg\, s\sin\alpha + \lambda R\theta - \lambda s$$

interpret

$$m\ddot{s} = -mg\sin\alpha - \lambda$$

$$\left.\begin{aligned} I\ddot{\theta} &= \lambda R \\ s &= R\theta \end{aligned}\right\} \lambda = \frac{I}{R}\ddot{\theta} = \frac{I}{R^2}\ddot{s}$$

$$m\ddot{s} = -mg\sin\alpha - \frac{I}{R^2}\ddot{s}$$

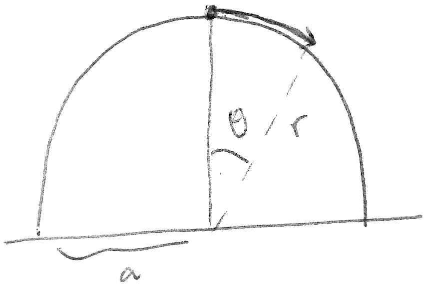
$$\left(m + \frac{I}{R^2}\right)\ddot{s} = -mg\sin\alpha \quad ✓$$

$\lambda \frac{\partial f}{\partial s} = -\lambda =$ ~~f~~ frictional force of constraint

$\lambda \frac{\partial f}{\partial \theta} = \lambda R =$ torque of constraint..

JJ

Particle on hemisphere



Constraint: $f(r) = r - a = 0$

(actually $f(r) \geq 0$)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$$

$$L' = L + \lambda(r - a)$$

$$\frac{\partial L'}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L'}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta + \lambda$$

$$m\ddot{r} = mr\dot{\theta}^2 - mg\cos\theta + \lambda$$

$$\frac{\partial L'}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial L'}{\partial \theta} = mgr\sin\theta$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = mgr\sin\theta$$

$$r - a = 0 \Rightarrow \begin{cases} r = a \\ \dot{r} = \ddot{r} = 0 \end{cases}$$

$$ma^2\ddot{\theta} = mga\sin\theta$$

$$\ddot{\theta} = \frac{g}{a}\sin\theta$$

$$\frac{1}{2}\dot{\theta}^2\Big| = -\frac{g}{a}\cos\theta\Big|$$

$$\frac{1}{2}\dot{\theta}^2 = \frac{g}{a}(1 - \cos\theta)$$

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

check $\theta = 0$

λ = normal force $\Rightarrow$ λ must > 0

~~Suppose $\lambda > 0$~~

$$\lambda = -ma\dot{\theta}^2 + mg\cos\theta$$

$$= \cancel{-\frac{2mg}{a}(1-\cos\theta)}$$

$$= -2mg(1 - \cos\theta) + mg\cos\theta$$

$$= mg(3\cos\theta - 2) > 0 \quad \text{if } \theta < \theta_0$$

$$\cos\theta_0 = \frac{2}{3}, \quad \theta_0 \approx 48^\circ$$