– Laws of motion, Conservation Laws

- Newton's 3 laws of motion
- Inertial reference frames, Galilean transformation
    - Non Cartesian coordinate systems (polar)
    - Velocity dependent forces
        - air resistance
        - Lorentz force
- Conservation of ~~m~~ linear momentum of a sys
    - Gravitational force ~~Gravity~~ between extended objects
    - Rocket motion
- Conservation of angular momentum
- Conservative ~~of energy~~ forces
    - ~~One~~ One dimension: position dependent force; period of motion
        - eqlbm: stable, unstable
    - Three dimension: conservative forces
- ~~Ene~~ Conservat of energy of a sys
- Gravitatal potential
    - of shell, of sphere

204 Searles
725-3625
naculich @ polar

Marion:

Grading
problem sets: 40%
1 mid: 25%
1 final: 35%

Outline of course

~~Newton~~

- ~~Newton's laws~~ Dynamics of System
- Conservation laws
- Lagrangian & Hamiltonian dynamics
- central force motion
- non inertial ref. frames
- rigid body motion
- coupled oscillators

Newton's Three Laws of Motion

- System of particles $i = 1, 2, \ldots$
- In an inertial reference frame:

$$2) \quad \frac{d\vec{p}_i}{dt} = \vec{F}_i = \sum_j \vec{F}_{ij}$$

$\vec{p}_i$ = momentum of i^{th} particle

$\vec{F}_{ij}$ = force exerted on i by j

$\vec{r}_i$ = position of i

Nonrelativistic: $\vec{p}_i = m_i \frac{d\vec{r}_i}{dt} = m_i \dot{\vec{r}}_i$

$$\vec{F}_i = m_i \ddot{\vec{r}}_i$$

1) Special case of 2^{nd} law

$$\vec{F}_i = 0 \implies \vec{p}_i = \text{const}$$

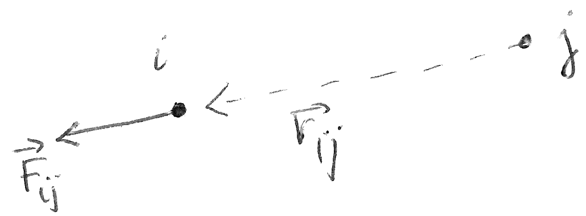
$$\vec{r}_i(t) = \vec{r}_i(0) + \frac{\vec{p}_i}{m} t$$

(at rest; in motion)

<u>Def</u> central force $\vec{F}_{ij} = f(r_{ij})\,\vec{r}_{ij}$

$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$

$r_{ij} = |\vec{r}_{ij}|$

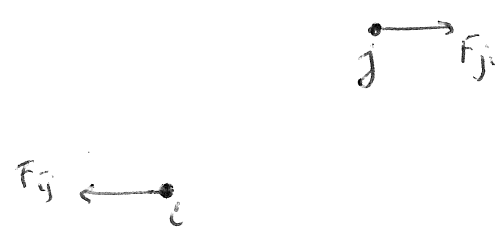


(e.g. gravity, Coulomb)

3) Action-reaction $\vec{F}_{ij} = -\vec{F}_{ji}$

a) strong form: forces collinear (obeyed only by central forces)

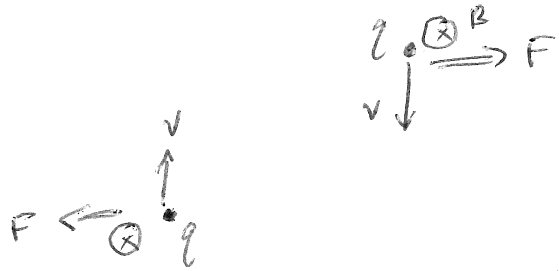
b) weak form: forces antiparallel but not collinear



Velocity dependent forces do not nec. obey 3rd law (e.g. magnetic)

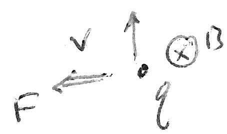
(also spin dependent potentials)

(((eg. Biot-Savart $\vec{B} = q \frac{\vec{v} \times \vec{r}}{|r|^3}$))))



(weak)

(neither)



((Problem is: assuming force at a distance; neglecting fields (which carry momentum etc)) → violates spec rel
→ magnetic.

((other limitations:
3rd law corresponds ~~Newton applies~~ to action-at-a-distance.

Not true, eg ~~if earth~~ two planets orbiting each other & one takes off at the speed of light, the first still experiences the old force whereas the other one experiences a different force))

((But we'll use 3rd law, but be aware of limitations))

Inertial reference frame

~~no fictitious forces (centrifugal, Coriolis)~~

[How define IRF?
Frame in which Newton's laws hold!
Tautology.
Nontrivial that such frames exist.
1st law has meaning as defining IRF.
~~But ... say that~~
Suppose in non-IRF, could blame motion
on imagined forces — centrifugal ...

no fictitious force
(centrifugal, Coriolis)

[How tell if they are fictitious?]
Don't obey 3rd law.

⟷

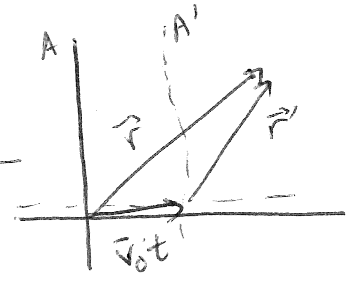
[Suppose we have an IRF

If A is an IRF, then any A′ moving
w/ const vel. w.r.t. F is an IRF

[i.e. Newton's laws obeyed in A′ too)

$\vec{r}$ = position of a pcl. w.r.t. A

$\vec{r}\,'$ = " " " " " A′



Let origins ~~of A and A′~~ A & A′ coincide at $t=0$

$$\vec{r}\,' = \vec{r} - \vec{v}_0 t$$
$$t' = t$$
} Galilean tx

[assume time is same in both coord sys]

[opposed to Lorentz tx

$$\dot{\vec{r}}' = \dot{\vec{r}} - \vec{v}_0$$
$$\ddot{\vec{r}}' = \ddot{\vec{r}}$$

1st law

$$\vec{F} = 0 \Rightarrow m\ddot{\vec{r}} = 0$$
$$\Downarrow$$
$$\vec{F}' = 0 \qquad m\ddot{\vec{r}}' = 0$$

2nd law

Let $\vec{F} = \vec{F}(\vec{r}_i - \vec{r}_j)$

$$\vec{r}_i' - \vec{r}_j' = \vec{r}_i - \vec{r}_j$$
$$\vec{F}' = \vec{F} = m\ddot{\vec{r}} = m\ddot{\vec{r}}'$$
$$\text{so } \vec{F}' = m\ddot{\vec{r}}'$$

Coordinate systems

$$\vec{F} = m\ddot{\vec{r}}$$

[a vector eqn, must resolve into components]

Cartesian

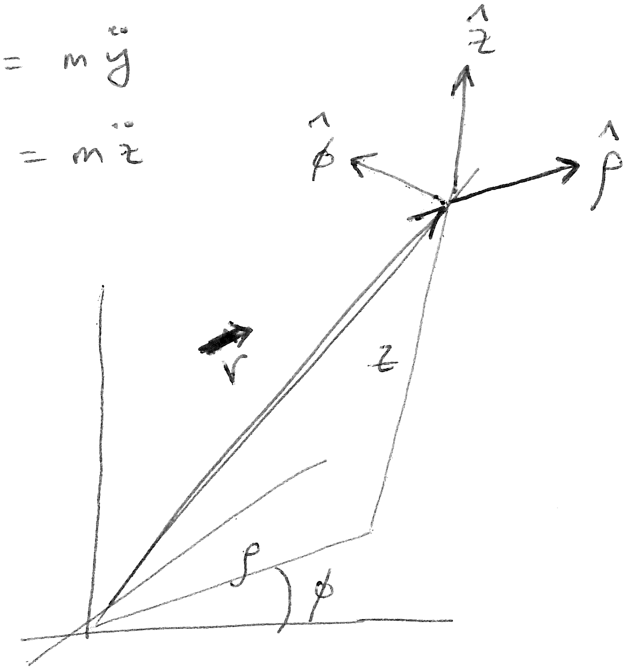
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases}$$

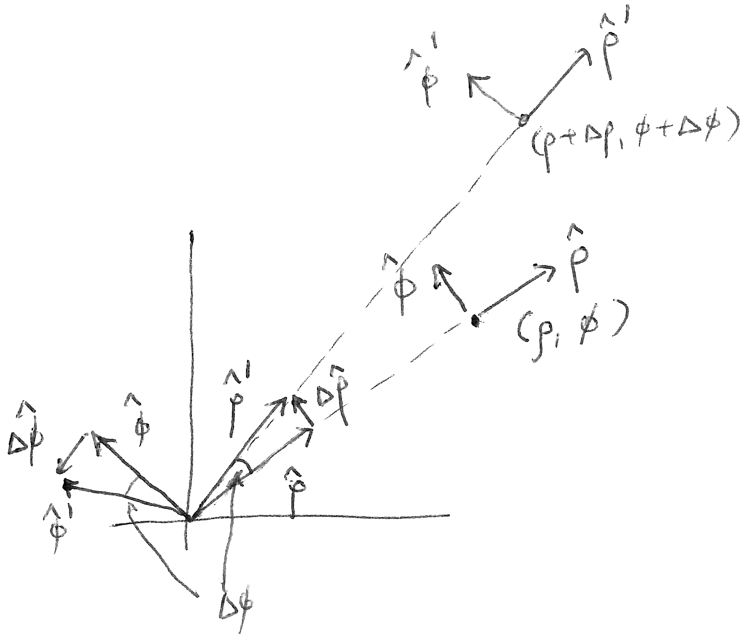
Cylindrical



$$\vec{r} = \rho\hat{\rho} + z\hat{z} \qquad (\text{no } \phi\hat{\phi} \text{ !})$$

$$\dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} + \dot{z}\hat{z} + z\dot{\hat{z}}$$

[because unit vectors $\hat{\rho}$ depends on ρ, ϕ, so if ρ, ϕ change, so does $\hat{\rho}$]



$$\begin{cases} \Delta\hat{\rho} = \hat{\phi}\,\Delta\phi \\ \Delta\hat{\phi} = -\hat{\rho}\,\Delta\phi \\ \Delta\hat{z} = 0 \end{cases}$$

$$\begin{cases} \dot{\hat{\rho}} = \dot{\phi}\,\hat{\phi} \\ \dot{\hat{\phi}} = -\dot{\phi}\,\hat{\rho} \\ \dot{\hat{z}} = 0 \end{cases}$$

$$\dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

$$\ddot{\vec{r}} = \ddot{\rho}\hat{\rho} + \dot{\rho}(\dot{\phi}\hat{\phi}) + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}(-\dot{\phi}\hat{\rho}) + \ddot{z}\hat{z}$$

$$= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

$$\vec{F} = F_\rho\hat{\rho} + F_\phi\hat{\phi} + F_z\hat{z}$$

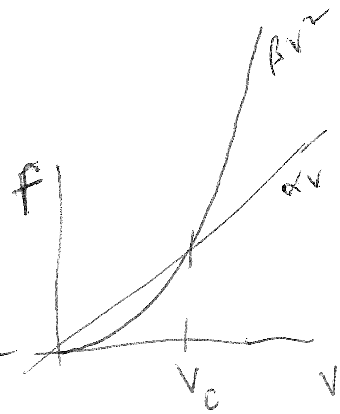
$$\begin{cases} F_\rho = m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\ F_z = m\ddot{z} \end{cases}$$

Velocity dependent forces

I. Air resistance

$$\vec{F} = -f(v)\hat{v}$$

For $v \ll v_{sound}$, $f(v) = \underbrace{\alpha v}_{\text{viscosity}} + \underbrace{\beta v^2}_{\text{turbulence}}$



∴

$v \ll v_c$: $f(v) \sim \alpha v$

$v \gg v_c$: $f(v) \sim \beta v^2$

$$\alpha v_c = \beta v_c^2$$

$$v_c = \frac{\alpha}{\beta}$$

For sphere of radius r

$\alpha = c_1 r$ $\qquad c_1 = 3.1\ E-3\ \frac{g}{cm\text{-}s}$

$\beta = c_2 r^2$ $\qquad c_2 = 8.7\ E-4\ \frac{g}{cm^3}$

$$v_c = \frac{c_1}{c_2 r}$$

B2

~~Velocity dependent forces~~

~~Air resistance~~

~~$\vec{F} = -f(v)\hat{v}$~~

~~$f(v) = (\alpha v + \beta v^2)$ $\quad v < v_{sound}$~~

~~For a sphere $\quad \alpha = c_1 r \quad c_1 = 3.1\text{E}{-3} \quad \beta = c_2 r^2 \quad c_2 =$~~

Assume $f(v) \approx \alpha v$

Moving object

$$m \frac{d^2x}{dt^2} = -\alpha \frac{dx}{dt}$$

((1st order bc doesn't depend on x))

$$m \frac{dv}{dt} = -\alpha v$$

((separable))

$$\frac{dv}{v} = -\frac{\alpha}{m} t$$

$$\ln v = -\frac{\alpha}{m} t + C$$

$v(0) = C$

$$v = C e^{-\frac{\alpha}{m} t}$$

$$v = v_0 e^{-\frac{\alpha t}{m}}$$

$$dx = v_0 e^{-\frac{\alpha t}{m}} dt$$

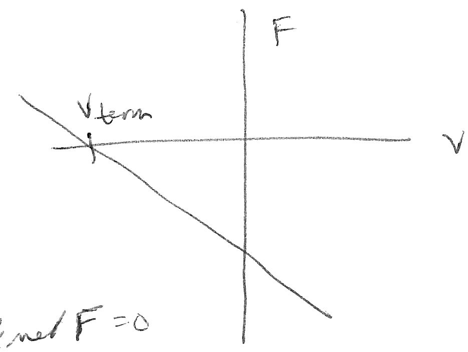
$$x - x_0 = -\frac{m v_0}{\alpha}\left(e^{-\frac{\alpha t}{m}} - 1\right)$$

$$x = x_0 + \frac{m v_0}{\alpha}\left(1 - e^{-\frac{\alpha t}{m}}\right)$$

((only goes a finite distance))

Falling object

$$m\frac{dv}{dt} = -\alpha v - mg = F$$



terminal velocity ~~[illegible]~~ F = 0

$$v_{term} = -\frac{mg}{\alpha}$$

Linear

$$m\frac{dv}{dt} + \alpha v = -mg$$

~~[illegible]~~

~~[illegible]~~

$$v = v_1 + v_2$$

v_1: specific, v_2: homogeneous

$$v_1 = v_{term}$$

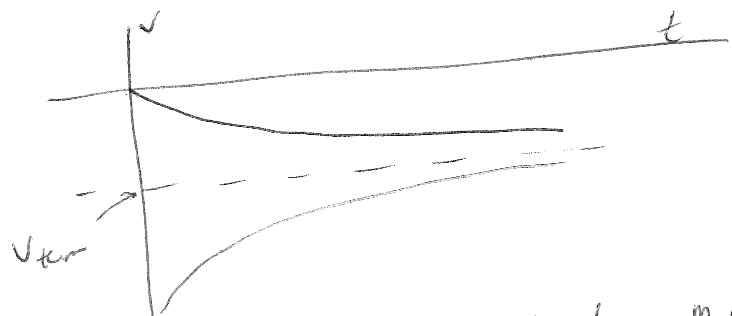
$$m\frac{dv_2}{dt} + \alpha v_2 = 0$$

$$v_2 = Ce^{-\frac{\alpha}{m}t}$$

$$v = v_{term} + Ce^{-\frac{\alpha}{m}t}$$

$$v(0) = v_{term} + C$$

$$v(t) = v_{term} + (v_0 - v_{term})e^{-\frac{\alpha}{m}t}$$



$$x(t) = x_0 + v_{term}t - \frac{m}{\alpha}(v_0 - v_{term})e^{-\frac{\alpha}{m}t}$$

B4

2. Lorentz force on charged particle

$$\vec{F} = q\vec{E} + \frac{q}{c}\vec{V}\times\vec{B}$$

Let $\vec{E}=0$, $\vec{B} = B_z\hat{z}$ = const

$$\vec{V}\times\vec{B} = (v_y B_z)\hat{i} + (-v_x B)\hat{j}$$

$$\frac{d\vec{v}}{dt} = \frac{q}{mc}\vec{V}\times\vec{B}$$

$$\dot{v}_x = \frac{qB_z}{mc} v_y$$

$$\dot{v}_y = -\frac{qB}{mc} v_x$$

$$\dot{v}_z = 0$$

$$v_z = v_{z0}$$

$$z = v_{z0} t + z_0$$

$$\ddot{v}_x = \frac{qB_z}{mc}\dot{v}_y = -\left(\frac{qB}{mc}\right)^2 v_x$$

$$\ddot{v}_y = -\left(\frac{qB}{mc}\right)^2 v_y$$

set $\phi = 0$

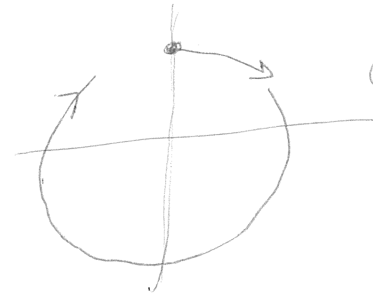
$$v_x = v_{x0}\cos(\omega t), \quad \omega = \frac{qB}{mc}$$

$$x = \frac{v_{x0}}{\omega}\sin(\omega t)$$

$$v_y = \frac{mc}{qB}\dot{v}_x = -v_{x0}\sin(\omega t)$$

$$y = +\frac{v_{x0}}{\omega}\cos(\omega t)$$

$$x = \frac{v_{x0}\, m c}{q B} \sin(\omega t)$$



$\odot B$

$$y = \frac{v_{x0}\, m c}{q B} \cos(\omega t)$$

$$x^2 + y^2 = \left(\frac{v_{x0}\, m c}{q B}\right)^2$$

circular motion

[$v_{x0} \uparrow$, $m \uparrow$ bigger

$B \uparrow$ tighter

~~$z = v_{z0} t + z_0$ helical motion~~

cyclotron frequency $\omega_c = \frac{qB}{mc}$

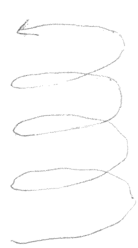
• indep of v ✓ [⇒ cyclotron]

[explain: use RF fields to ↑ velocity...]

[moves in wider circles but same freq.]

$z = v_{z0} t + z_0$

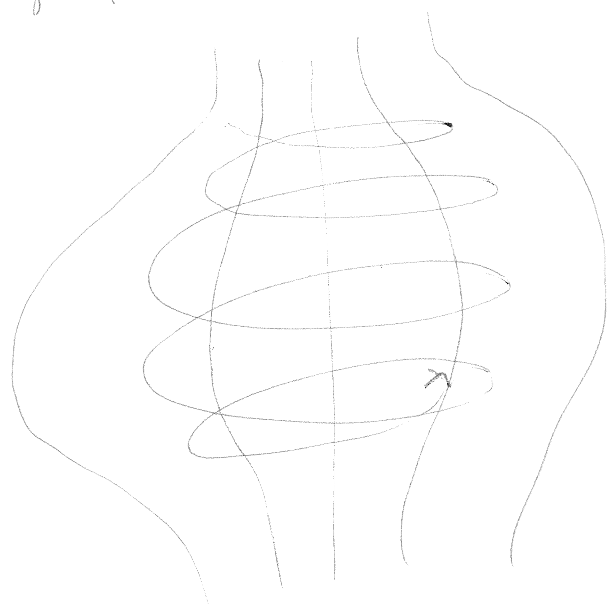
helical motion



[short discussion of $E \times B$ drift]

Inhomogeneous field

Higher field reflects particle



Magnetic bottle

Conservation of linear momentum of a system

System of particles of $\begin{cases} \text{mass } m_i \\ \text{position } \vec{r}_i \\ \text{momentum } \vec{p}_i \end{cases}$

Total momentum $\vec{P} = \sum_i \vec{p}_i$

Force on i $= \vec{F}_i = \vec{F}_i^{ext} + \sum_{\substack{j \\ (j \neq i)}} \vec{F}_{ij}$

((no self-force))

$$\frac{d\vec{P}}{dt} = \sum_i \frac{d\vec{p}_i}{dt} = \underbrace{\sum_i \vec{F}_i^{ext}}_{\vec{F}^{ext}} + \sum_{\substack{i,j \\ (j \neq i)}} \vec{F}_{ij}$$

Assume internal forces obey weak form of 3rd law.

$$\sum_{\substack{i,j \\ (i \neq j)}} \vec{F}_{ij} = \sum_{\substack{i,j \\ (i<j)}} \vec{F}_{ij} + \underbrace{\sum_{\substack{i,j \\ (i>j)}} \vec{F}_{ij}}_{\sum_{\substack{j,i \\ (j>i)}} \vec{F}_{ji}} = \sum_{\substack{i,j \\ (i<j)}} \underbrace{(\vec{F}_{ij} + \vec{F}_{ji})}_{0} = 0$$

$$\boxed{\frac{d\vec{P}}{dt} = \vec{F}^{ext}} \qquad \vec{F}^{ext} = 0 \Rightarrow \vec{P} = \text{const}$$

- In the absence of external forces, the total linear momentum of a system is conserved

Center of mass $\vec{R} = \dfrac{\sum m_i \vec{r}_i}{M}$

$$M = \sum m_i$$

$$\sum m_i \vec{r}_i = M\vec{R}$$

Continuous system $\vec{R} = \dfrac{\int \vec{r}\, dm}{M}$

Non relativistic $\vec{P} = \sum m_i \dot{\vec{r}}_i = M\dot{\vec{R}}$

- Total linear momentum of a system ~~equals~~ is equal to that of a single particle of mass M moving with the velocity of the center of mass

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{F}^{ext}$$

- Center of mass moves as if it were a single particle of mass M acted on the total external force.

Gravitation

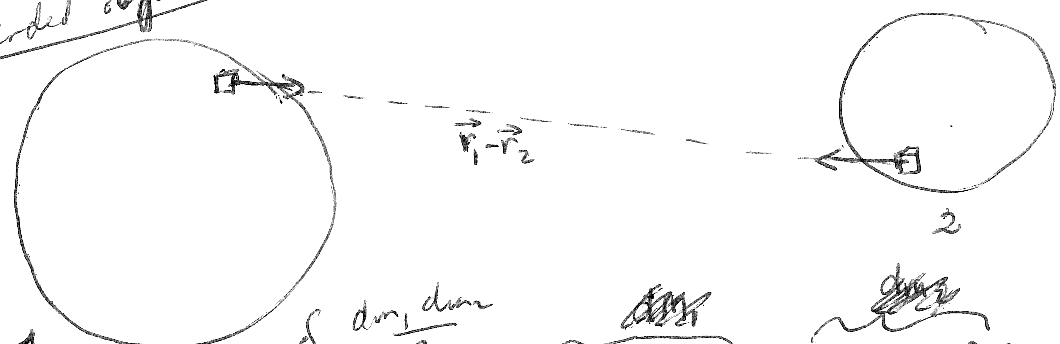
$$\vec{F}_{ij} = -\frac{G m_1 m_2}{r_{ij}^2}\hat{r}_{ij}$$

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

$$r_{ij} = |\vec{r}_{ij}|$$

$$\hat{r}_{ij} = \frac{\vec{r}_{ij}}{r_{ij}}$$ ((for point masses))

Extended objects



1

$$\vec{F}_{12} = -G\int \frac{dm_1\, dm_2}{r_{12}^2}$$

$$\vec{F} = -G\int \frac{\overbrace{\rho_1(\vec{r}_1)\, d^3\vec{r}_1}^{dm_1}\ \overbrace{\rho_2(\vec{r}_2)\, d^3\vec{r}_2}^{dm_2}}{r_{12}^2}\hat{r}_{12}$$

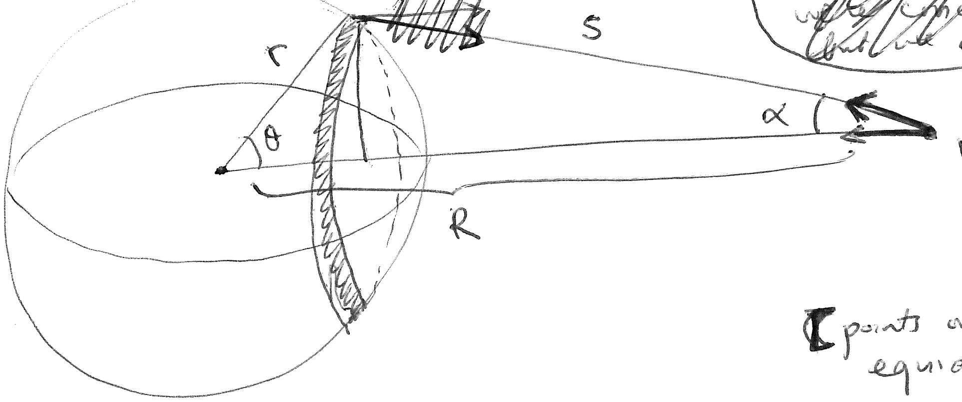
[Body 1 internal forces cancel out, so cm moves acc. to $\vec{F}_{ext}$ but must calc $\vec{F}_{ext}$]

[double integral, but we'll use single integral + then some careful argumentation]

Force exerted on mass m by a shell of radius r

~~Force exerted on a mass shell of radius r by point mass m~~

~~[we know that net force acts as if all the mass were concentrated at center but we don't know the shape of the forces]~~



[points on a ring are equidistant from m]

[of course Force vectors point in diff directions]

$$F = \frac{G m \rho \, dV}{s^2} \cos\alpha$$

$$dV = dr \;\; r\,d\theta \;\; 2\pi r \sin\theta = 2\pi r^2 dr \sin\theta \, d\theta$$

[thickness of shell, width of ring, circumference]

$$\rho = \frac{dM}{4\pi r^2 dr} \qquad dM = \text{mass of shell}$$

$$F = \frac{G(dM)m}{2} \int_0^{\pi} \frac{\sin\theta \, d\theta \, \cos\alpha}{s^2}$$

see my 300 calculation

~~scribbled out~~

$[s, \theta, \alpha]$
[solve in terms of s]

$$R = r\cos\theta + s\cos\alpha$$

[[check for $\theta > \pi$]]

$$r\sin\theta = s\sin\alpha$$

$$r^2 = s^2\sin^2\alpha + (R - s\cos\alpha)^2$$

$$= s^2 + R^2 - 2Rs\cos\alpha$$

$$\cos\alpha = \frac{R^2 - r^2 + s^2}{2Rs}$$

$$s^2 = r^2\sin^2\theta + (R - r\cos\theta)^2$$

$$= r^2 + R^2 - 2Rr\cos\theta$$

$$\cos\theta = \frac{R^2 + r^2 - s^2}{2Rr}$$

$$-\sin\theta\, d\theta = -\frac{s\,ds}{Rr}$$

$$F = \frac{G\,dM\,m}{2}\int_{R-r}^{R+r} \frac{s\,ds}{Rr}\,\frac{R^2 - r^2 + s^2}{2Rs}\,\frac{1}{s^2}$$

[[inside shell limits: $r-R$ to $r+R$ $\Rightarrow F = 0$]]

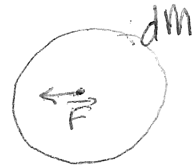
$$= \frac{G\,dM\,m}{4R^2 r}\int\left(\frac{R^2 - r^2}{s^2} + 1\right)ds$$

$$\underbrace{\left[-\frac{R^2 - r^2}{s} + s\right]_{R-r}^{R+r}}_{4r}$$

$$= \frac{G\,dM\,m}{R^2}$$

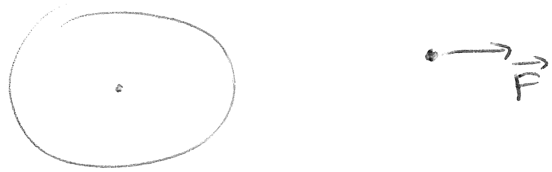
[as if mass were all at the center]]
in terms of strength,
not obvious

sphere: $F = \int \frac{G\,dM\,m}{R^2} = \frac{GMm}{R^2}$

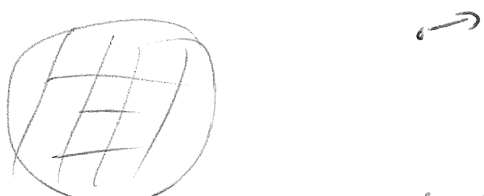


• force of particle on shell = $\frac{G(dM)m}{R^2}$

• ~~forces~~
• 3rd law $\Rightarrow$ force of ~~particle~~ shell on particle is $\frac{G(dM)m}{R^2}$



$\Rightarrow$ force of sphere on particle is $\frac{GMm}{R^2}$



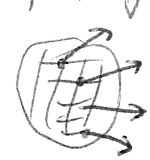
so replace sphere by particle at center

equiv

• Consider 2 sphere

For each point of 2, replace 1 by a point

equiv



By prior calc, force of particle on sphere is same as force of particle on a particle located at its center

[In another situat,

force acts on cm

but magnitude of force is
not the same as if all
mass was concentrated at cm

not same force as

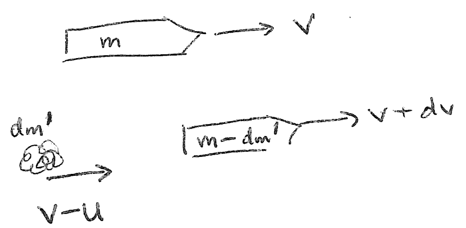
[NB This is an important pt to stress,
students think that all collect of masses
can be replaced by mass at cm.
NOT just spherical shells]

Rocket motion

Free space

$\vec{F}^{ext} = 0 \Rightarrow \vec{p}$ conserved

t



$t + dt$

$dm' > 0$

u = exhaust vel rel. to rocket

$$p(t) = p(t+dt)$$

$$mv = (m - dm')(v + dv) + dm'(v-u)$$

$$\cancel{mv} = \cancel{mv} - \cancel{dm'v} + m\,dv - \underbrace{dm'\,dv}_{O(\Delta^2)} + \cancel{dm'v} - dm'u$$

$$0 = m\,dv - dm'u$$

$$m\frac{dv}{dt} = \underbrace{u\frac{dm'}{dt}}_{\text{thrust}} = -u\frac{dm}{dt} > 0$$

$$\frac{dm}{dt} < 0$$

$$dm = -dm'$$

$$\int_0^t \frac{dv}{dt}\,dt = -\int_0^t \frac{u}{m}\frac{dm}{dt}\,dt$$ assume u = const

$$v(t) - v_0 = -u\ln\left(\frac{m(t)}{m_0}\right)$$

$$v(t) = v_0 + u\ln\left(\frac{m_0}{m(t)}\right)$$

~~m(t) = payload~~

~~M_0~~

To find $x(t)$, need another ~~eq~~ relation:

[y? 3 vbls v, t, m]

$$\frac{dm}{dt} = -\alpha = \text{const} \qquad \text{thrust} = +\alpha u$$

$$m = m_0 - \alpha t$$

$$\frac{dx}{dt} = v(t) = v_0 + u \ln\left(\frac{m_0}{m_0 - \alpha t}\right)$$

$$= v_0 - u \ln\left(1 - \frac{\alpha}{m_0} t\right)$$

$$\cancel{x(t) = x_0 + v_0 t + \frac{m_0 u}{\alpha}\left(1 - \frac{\alpha}{m_0}t\right) \ln\left(1 - \frac{\alpha}{m_0}t\right) - \frac{m_0 u}{\alpha}\left(1 - \frac{\alpha}{m_0}t\right)}$$

~~Integrate to get~~ x

$$x(t) - x_0 = \int_0^t dt \left[v_0 - u \ln\left(1 - \frac{\alpha}{m_0} t\right)\right]$$

If $v_0 = 0$, $\quad v_f = u \ln\left(\frac{m_0}{m_f}\right)$, $\quad m_0 = m_f + m_{fuel}$

$u_{max} \approx 5000 \frac{m}{s}$ for chemical processes

(in reality $u_{max} \approx 2500 \frac{m}{s}$)

~~$V_{escape} \approx 11000 \frac{m}{s}$~~

v_f	$\frac{m_0}{m_f}$	$\frac{m_{fuel}}{m_f}$
u	2.7	1.7
$2u$	7.4	~~8.4~~ 6.4
$4u$	54.5	53.5

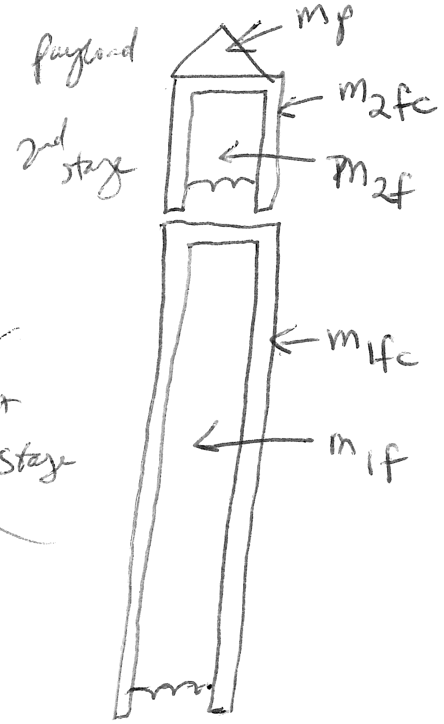
$V_{escape} \approx 11000 \frac{m}{s}$

[so need $\frac{m_{fuel}}{m_f} \approx 80$, neglecting gravity!]

[$\frac{m_{fuel}}{m_{rocket}}$ limited by structural constraints]

$\max\left(\frac{m_{fuel}}{m_{rocket}}\right) \sim 10-20$

Multistage rockets



2 stage

v_1 = velocity after 1st stage

$$v_1 = V_0 + u \ln\left(\frac{m_p + m_{2fc} + m_{2f} + m_{1fc} + m_{1f}}{m_p + m_{2fc} + m_{2f} + m_{1fc}}\right) \quad \text{1st stage}$$

Jettison stage 1 fuel container

$$V_2 = v_1 + u \ln\left(\frac{m_p + m_{2fc} + m_{2f}}{m_p + m_{2fc}}\right)$$

$$= v_0 + u \ln\left(\frac{m_p + m_{2fc} + m_{2f}}{m_p + m_{2fc}} \; \frac{m_p + m_{2fc} + m_{2f} + m_{1fc} + m_{1f}}{m_p + m_{2fc} + m_{2f} + m_{1fc}}\right)$$

which is greater than

$$V_0 + u \ln\left(\frac{m_p + m_{2fc} + m_{2f} + m_{1fc} + m_{1f}}{m_p + m_{2fc} + m_{1fc}}\right) \leftarrow \text{single stage}$$

by a factor of

$$\left(\frac{(m_p + m_{2fc}) + m_{2f}}{(m_p + m_{2fc})}\right)\left[\frac{(m_p + m_{2fc} + m_{1fc})}{(m_p + m_{2fc} + m_{1fc}) + m_{2f}}\right] > 1$$

≥ 1 $\qquad$ < 1

DT

E.g.

$$\left.\begin{array}{l} m_p = 1 \\ m_{2fc} = 1 \\ m_{2f} = 8 \end{array}\right\} 10$$

$m_{1fc} = 10$ 100

$m_{1f} = 80$

<u>2 stages</u>

$$v_f = u \ln\left(\frac{100}{20}\right) + u \ln\left(\frac{10}{2}\right) = 2u \ln 5 \approx 3.2\,u$$

<u>1 stage</u>

$$v_f = u \ln\left(\frac{100}{12}\right) \approx 2.1\,u$$

(50% more)

Rocket ascend under gravity

$$F = \frac{dP}{dt}$$

$$F dt = dP = P(t+dt) - P(t)$$

$$-mg\,dt = m\,dv + u\,dm$$

$$-g\,dt = dv + \frac{u\,dm}{m}$$

3 variables (m, v, t), need another eqn

$$\frac{dm}{dt} = -\alpha$$

$$dm = -\alpha\,dt$$

$$m = m_0 - \alpha t$$

Elim t:

$$\frac{g}{\alpha}\,dm = dv + \frac{u\,dm}{m}$$

$$\int_0^v dv = \int_{m_0}^{m} \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm$$

$$v = \frac{g}{\alpha}(m - m_0) - u\ln\left(\frac{m}{m_0}\right)$$

$$= -gt + u\ln\left(\frac{m_0}{m}\right)$$

$$= -gt + u\ln\left(\frac{m_0}{m_0 - \alpha t}\right)$$

$$x = \int [\quad '' \quad]\,dt$$

NB. Thrust > gravity for liftoff

$$u \frac{dm}{dt} > mg$$

Saturn 5: mg at liftoff = 6.2 millions lbs.

~~total~~ thrust = 7.6 " lbs

[after liftoff mg decreases, so more of thrust used for acceleration]

Conservation of angular momentum of a system

$\vec{L}_i = \vec{r}_i \times \vec{p}_i$

$$\frac{d\vec{L}_i}{dt} = \underbrace{\dot{\vec{r}}_i \times \vec{p}_i}_{0} + \vec{r}_i \times \dot{\vec{p}}_i = \vec{r}_i \times \vec{F}_i = \vec{N}_i = \text{torque on } i$$

Total angular momentum $\vec{L} = \sum \vec{L}_i$

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i = \underbrace{\sum_i \underbrace{\vec{r}_i \times \vec{F}_i^{ext}}_{\vec{N}_i^{ext}}}_{\vec{N}^{ext}} + \sum_{\substack{ij \\ (j \neq i)}} \vec{r}_i \times \vec{F}_{ij}$$

$$\sum_{\substack{ij \\ (j \neq i)}} \vec{r}_i \times \vec{F}_{ij} = \sum_{i<j} \vec{r}_i \times \vec{F}_{ij} + \underbrace{\sum_{i>j} \vec{r}_i \times \vec{F}_{ij}}_{\sum_{j>i} \vec{r}_j \times \underbrace{\vec{F}_{ji}}_{-\vec{F}_{ij} \ (3^{rd}\ \text{law})}}$$

$$= \sum_{i<j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}$$

$$= \sum_{i<j} \vec{r}_{ij} \times \vec{F}_{ij}$$

If internal force are central (obey strong form of 3^{rd} law)

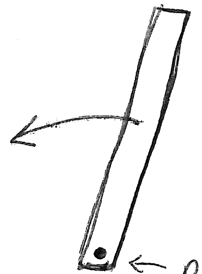
$\vec{F}_{ij} \propto \vec{r}_{ij}$, $\vec{r}_{ij} \times \vec{F}_{ij} = 0$, ~~then internal~~

then sum of internal torques vanishes.

$$\frac{d\vec{L}}{dt} = \vec{N}^{ext} \qquad \vec{N}^{ext} = 0 \Rightarrow \vec{L} = \text{const}$$

- In the absence of external torques, the total angular momentum of a system is conserved [although our proof of cons of mom + cons of angular moment depended on 3rd law in weak or strong forms these cons laws are more general. when forces don't obey 3rd law (eg magnetic), it is psbl to modify def of mom ~~so~~ so that total is cons'd, ↓ this def will incl mom + ang moment of the fields as well as particles]

E3



← Pivot exerts force $\Rightarrow \vec{P}$ not conserved

$\vec{F}^{ext} \neq 0$

$\vec{L}$ about pivot

$\cancel{\frac{d\vec{L}}{dt}} = \vec{N}^{ext} = \vec{r} \times \vec{F} = 0 \Rightarrow \vec{L}$ conserved about pivot

(Also about cm b/c $\vec{F} \parallel \hat{r} \Rightarrow \vec{N} = 0$)

~~Conservation of angular momentum of a system~~

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i = m_i(\vec{r}_i \times \dot{\vec{r}}_i)$$

((N.B. $\vec{L}$ depends on origin of coords!))

Let $\vec{r}_i'$ denote position of i relative to center of mass

$$\vec{r}_i' = \vec{r}_i - \vec{R}$$

$$\vec{r}_i = \vec{R} + \vec{r}_i'$$

Total angular momentum $\vec{L} = \sum_i \vec{L}_i$

$$\cancel{= \sum_i (\vec{R} + \vec{r}_i') \times m_i(\dot{\vec{R}} + \dot{\vec{r}}_i')}$$

$$= \sum_i m_i \left[(\vec{R} + \vec{r}_i') \times (\dot{\vec{R}} + \dot{\vec{r}}_i') \right]$$

$$= \sum_i m_i \left[(\vec{R} \times \dot{\vec{R}}) + (\vec{r}_i' \times \dot{\vec{R}}) + (\vec{R} \times \dot{\vec{r}}_i') + (\vec{r}_i' \times \dot{\vec{r}}_i') \right]$$

$$= \vec{R} \times M\dot{\vec{R}} + \underbrace{\left(\sum m_i \vec{r}_i'\right)}_{0} \times \dot{\vec{R}} + \vec{R} \times \left(\sum m_i \dot{\vec{r}}_i'\right) + \sum (\vec{r}_i' \times m \dot{\vec{r}}_i')$$

$$\vec{R} \times \frac{d}{dt}\underbrace{\left(\sum m_i \vec{r}_i'\right)}_{0}$$

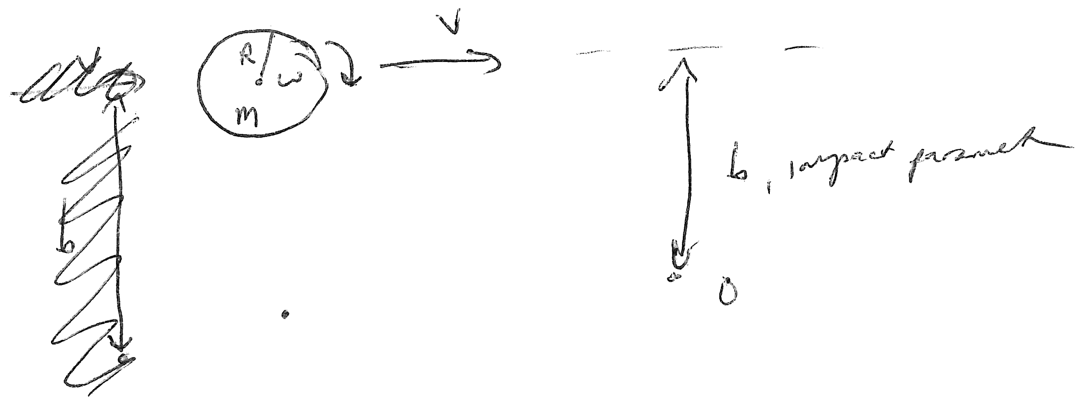
$$\sum m_i \vec{r}_i' = \sum m_i (\vec{r}_i - \vec{R}) = \sum m_i \vec{r}_i - M\vec{R} = 0$$

$$\vec{L} = \vec{R} \times \vec{P} + \sum (\vec{r}_i' \times \vec{p}_i')$$

[Total angular momentum of a system about some origin O is the sum of angular momentum of the center of mass about O and the total angular momentum of the system about its CM.

((iff $\vec{P} = 0$, then $\vec{L}$ indep of choice of origin))

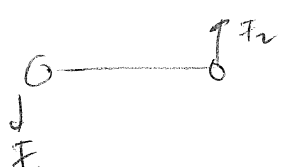
$$\vec{L} = \vec{R} \times \vec{P} + \vec{L}_{int}$$

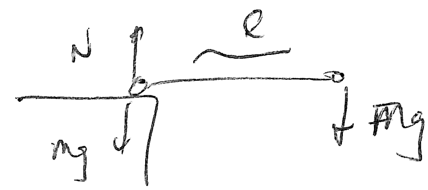


$$L = mvb + \left(\tfrac{1}{2}MR^2\right)\omega$$

~~Th~~ no

$$\frac{d\vec{L}}{dt} = \vec{R} \times \vec{F}^{ext} + \frac{d\vec{L}_{int}}{dt} = \vec{N}^{ext}$$





$F^{ext} = 0$

$N^{ext} \neq 0$

$F^{ext} = mg$

$N^{ext} = lmg$

$R \times F^{ext} = \frac{l}{2} mg$

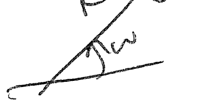
so $\frac{1}{2}$ goes into translat

$\frac{1}{2}$ into rotat

this ended up beg confusing

If going to explain need go into greater detail

Parallel axes thm



$$L = I\omega = R(\underbrace{Mv}_{M\omega R}) + I_{cm}\omega = (MR^2 + I_{cm})\omega$$

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \vec{r}_{cm} \times \vec{p}_{cm} + \underbrace{\sum \vec{r}_i{}' \times \vec{p}_i{}'}_{\vec{L}' \text{ wrt. cm}}$$

$$\dot{\vec{L}} = \sum_i \underbrace{\dot{\vec{r}}_i \times \vec{p}_i}_{0} + \sum \vec{r}_i \times \dot{\vec{p}}_i$$

$$= \sum \vec{r}_i \times \left(\vec{F}_i^{ext} + \sum_j F_{ij}\right)$$

$$= \sum \vec{r}_i \times \vec{F}_i^{ext}$$

$$\dot{\vec{L}} = \vec{M}$$ ← holds for any origin of coords in an inertial ref frame

---

$$\vec{M} = \frac{d\vec{L}}{dt}$$

$$\sum \underbrace{\vec{r}_i}_{\vec{r}_{cm} + \vec{r}_i{}'} \times \vec{F}_i^{ext} = \frac{d}{dt}\left(\vec{r}_{cm} \times \vec{p}_{cm} + \vec{L}'\right)$$

$$\vec{r}_{cm} \times \sum \vec{F}_i^{ext} + \underset{\substack{\uparrow \\ \text{torque abt cm}}}{\vec{M}'} = \vec{r}_c \times \frac{d\vec{p}_{cm}}{dt} + \frac{d\vec{L}'}{dt}$$

$$\vec{M}' = \frac{d\vec{L}'}{dt}$$

N.B. if $\frac{d\vec{p}_{cm}}{dt} \neq 0$ then $\vec{r}_{cm}$ defines a noninertial reference frame!

holds true in a noninertial ref frame! abt cm! ← important

Forces dependent only on position

[not $\vec{v}$ or t]

$$\vec{F} = \vec{F}(\vec{r})$$

eg. harmonic osc.
central force

[not friction, why? bc $\vec{F} \propto \hat{v}$]

one-dimension:

$$m\ddot{x} = F(x)$$

Define $\quad U(x) = -\int_0^x F(x')\,dx'$

$$\frac{dU}{dx} = -F(x) \qquad \text{[why?]}$$

$$m\ddot{x} + \frac{dU}{dx} = 0$$

[trick for soln]

$$m\dot{x}\ddot{x} + \frac{dU}{dx}\dot{x} = 0 \qquad \frac{dU}{dt} = \frac{dU}{dx}\frac{dx}{dt}$$

$$m\frac{1}{2}\frac{d}{dt}\dot{x}^2 + \frac{dU}{dt} = 0$$

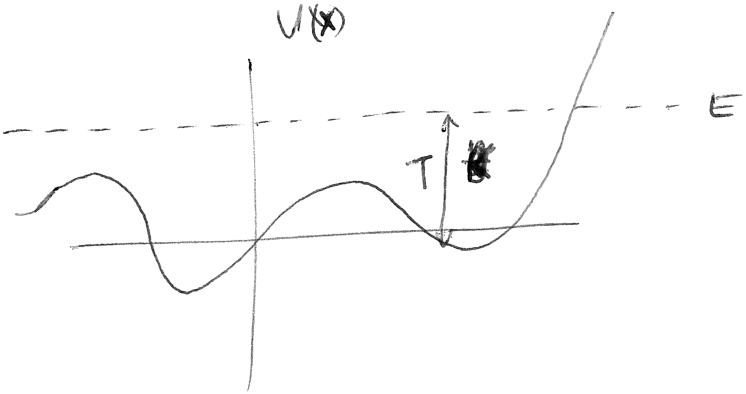
$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + U\right) = 0$$

$$\underbrace{\frac{1}{2}m\dot{x}^2}_{T} + U(x) = E$$

E fixed by initial conditions

$$E = \frac{1}{2}mv_0^2 + U(x_0)$$

$$T + U = E$$



$$\dot{x}^2 = \frac{2}{m}(E - U(x))$$

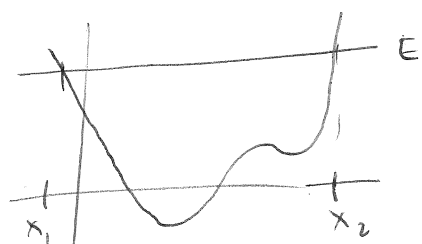
$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E - U(x))}$$

$$\int_{x_0}^{x} \frac{\pm\, dx}{\sqrt{\frac{2}{m}(E-U(x))}} = \int_{t_0}^{t} dt = t - t_0$$

Integrate, invert to get $x(t)$. [possible need numerical integration]

"Reduced to quadratures"

[quadrature = old word for integrals]



Turning points: $x \ni E = U(x)$

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}}$$

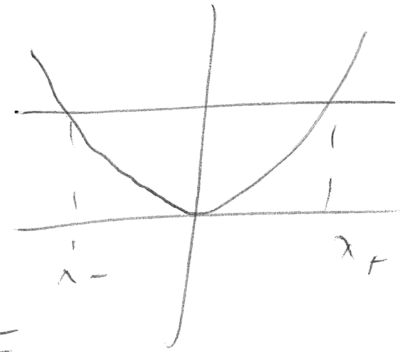
[particle can't go beyond t.p.; if $\exists$ 2, motion is bounded]

Period $\tau = 2(t_2 - t_1)$

do d.m.e. oscillator here

see ahead 2 pages

Harmonic oscillator



$U(x) = \frac{1}{2}kx^2$

turning points $\frac{1}{2}kx_\pm^2 = E$

$$x_\pm = \pm\sqrt{\frac{2E}{k}}$$

$$T = 2\int_{x_-}^{x_+} \frac{dx}{\sqrt{\frac{2}{m}(E - \frac{1}{2}kx^2)}}$$

$$= 4\int_0^{x_+} \frac{dx}{\sqrt{\frac{k}{m}(x_+^2 - x^2)}}$$

$$= 4\sqrt{\frac{m}{k}}\int_0^{x_+} \frac{dx}{\sqrt{x_+^2 - x^2}}$$

Does it diverge at $x = x_+$?

Check: $x = x_+ - \epsilon$

$dx = -d\epsilon$

$$\int^0 \frac{-d\epsilon}{\sqrt{\underbrace{x_+^2 - (x_+ - \epsilon)^2}_{2x_+\epsilon + O(\epsilon^2)}}}$$

$$\sim \int^0 \frac{d\epsilon}{\sqrt{\epsilon}} \sim \sqrt{\epsilon}\Big|^0 = 0 \qquad \underline{\underline{\text{ok}}}$$

F4

$$T = 4\sqrt{\frac{m}{k}}\int_0^{x_+}\frac{dx}{\sqrt{x_+^2 - x^2}}$$

$$x = x_+ \sin\theta$$

$$T = 4\sqrt{\frac{m}{k}}\underbrace{\int_0^{\frac{\pi}{2}}\frac{x_+\cos\theta\, d\theta}{x_+ \cancel{\cos\theta}\sqrt{1-\sin^2\theta}}}_{\frac{\pi}{2}}$$

$$= 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\omega} \checkmark$$

$$t = \sqrt{\frac{m}{k}}\int_{x_-}^{0}\frac{dx}{\sqrt{x_-^2 - x^2}}$$

$$x = x_- \sin\theta$$

$$t = \sqrt{\frac{m}{k}}\int_{\theta=\frac{\pi}{2}}^{\theta=\arcsin\frac{x}{x_-}} d\theta$$

$$= \sqrt{\frac{m}{k}}\left(\arcsin\left(\frac{x}{x_-}\right) - \frac{\pi}{2}\right)$$

$$x = x_- \underbrace{\sin\left(\omega t + \frac{\pi}{2}\right)}_{\cos(\omega t)}$$

(perhaps do the first as an example of exactly integrable motion)

[Lissajous figures]

[Problem: show that if turning point occurs at a local max of $U(x)$, then period is infinite]

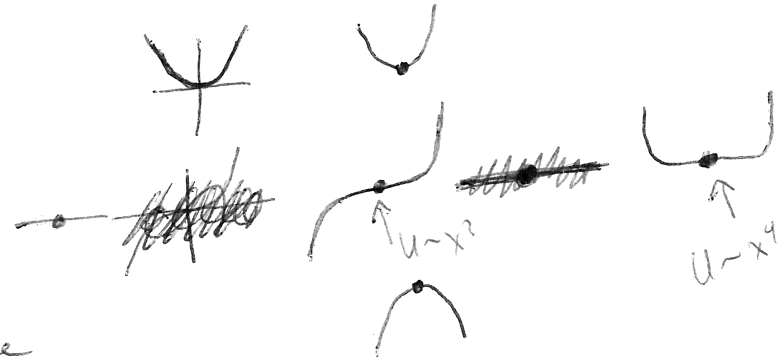
[correction to pendulum]

Equilibrium $\Rightarrow F = 0 \Rightarrow \frac{dU}{dx} = 0$, extrema of U

$\frac{d^2U}{dx^2} > 0$ stable

$\frac{d^2U}{dx^2} = 0$ neutral

$\frac{d^2U}{dx^2} < 0$ unstable



Small amplitude motion about minima

Let x_0 be a minimum

$$U(x) = U(x_0) + (x-x_0)\underbrace{U'(x_0)}_{0} + \frac{1}{2}(x-x_0)^2 \underbrace{U''(x_0)}_{\equiv k} + \dots$$

$$F(x) = -\frac{dU}{dx} = -k(x-x_0) + \dots$$

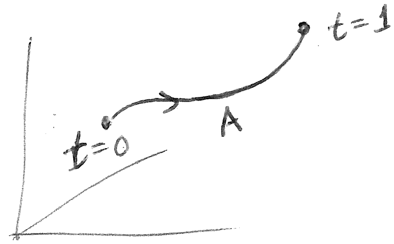
(harmonic oscillator)

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(x_0)}{m}}$$

3 dimensions

Choose a path A ~~$\vec{r}_A(t)$~~ parametrized by t, $\vec{r}_A(t)$

$0 \le t \le 1$



Work done on particle

$$W_{01}^{A} = \int_A \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}_A}{dt}\, dt = \int_0^1 \vec{F} \cdot \vec{v}\, dt$$

[force tangent to the motion]

Power generated $P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$

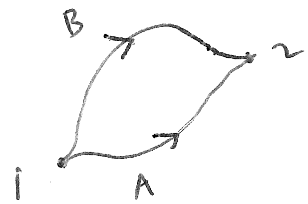
Forces of constraint } do no work because $\vec{F} \perp d\vec{r}$
Magnetic forces }

$$W_{01}^{A} = \int_0^1 m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt = \cancel{\int_0^1 \frac{1}{2} m \frac{d}{dt}(\vec{v}^2)\, dt}$$

$$= \int_0^1 \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v})\, dt$$

$$= \int_0^1 \frac{dT}{dt}\, dt = T(1) - T(0)$$

Conservative force $\equiv$ work done is independent of path



$$\underset{\text{along A}}{\int_1^2} \vec{F}\cdot d\vec{r} = \underset{\text{along B}}{\int_1^2} \vec{F}\cdot d\vec{r}$$

$$\underset{A}{\int_1^2} \vec{F}\cdot d\vec{r} + \underset{B}{\int_2^1} \vec{F}\cdot d\vec{r} = 0$$

$$\oint \vec{F}\cdot d\vec{r} = 0 \quad \text{for } \underline{\text{any}} \text{ closed path}$$

[eg. not friction or other dissipative force]

Stokes ~~[illegible]~~ $\oint \vec{F}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$

At this pt, students claimed not know about curl, or Stokes law, so I explained curl in physical terms + worked out its def in Cartes. coords + did a quick proof of Stokes

$$\int (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = 0 \quad \text{for any area}$$

Conservative $\Leftrightarrow$ $\vec{\nabla}\times\vec{F} = 0$

$\vec{F} = -\vec{\nabla}U \Rightarrow \vec{\nabla}\times\vec{F} = -\vec{\nabla}\times\vec{\nabla}U = 0$

If

Poincaré lemma: $\vec{\nabla}\times\vec{F} = 0 \Rightarrow \vec{F} = -\vec{\nabla}U$

~~(obvious $\vec{\nabla}\times\vec{\nabla}U = 0$)~~

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = -\int_1^2 \vec{\nabla}U\cdot d\vec{r} = -\int_1^2 dU = -U_2 + U_1$$

$$= \frac{dU}{dx}dx + \frac{dU}{dy}dy + \frac{dU}{dz}dz = dU, \text{ if } U = U(\vec{x}) \text{ } \underline{\text{only}}$$

$$W_{12} = T_2 - T_1 = -U_2 + U_1$$

$$T_2 + U_2 = T_1 + U_1$$

$$\underline{E = T + U \text{ is conserved}}$$

[mention? zero of energy is irrelevant]

~~If the~~

F8

- ~~[The total energy T+U of a particle acted on by conservative forces is conserved~~

- If the forces acting on a particle are conservative, then the total energy of the particle, $T+U$, is conserved

[in one dimension, any force that depends only on position is conservative]

Alternatively, <u>define</u> the potential energy to be

$$U(\vec{r}) = -\int_0^{\vec{r}} \vec{F} \cdot d\vec{r}$$

Since force is conservative, the path is irrelevant.

So U is single valued

#10

- Central forces are conservative

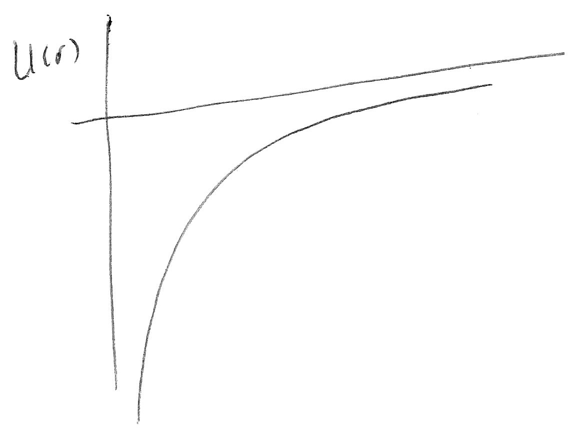
- Potential energy for central force $U = U(r)$

$$\vec{F} = -\vec{\nabla} U = -\frac{dU}{dr}\hat{r}$$

Gravitation $\vec{F} = -\frac{GMm}{r^2}\hat{r}$

$$U = -\frac{GMm}{r} + C$$

Let $U(\infty) = 0 \Rightarrow C = 0$



~~[illegible]~~ Energy of a system

$$T = \sum_i \frac{1}{2} m \dot{\vec{r}}_i^{\,2}$$

$$\vec{r}_i = \vec{R} + \vec{r}_i^{\,\prime}$$

$$T = \sum \frac{1}{2} m_i \left(\dot{\vec{R}}^2 + 2\dot{\vec{R}} \cdot \dot{\vec{r}}_i^{\,\prime} + \dot{\vec{r}}_i^{\,\prime 2} \right)$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \dot{\vec{R}} \underbrace{\sum m_i \dot{\vec{r}}_i^{\,\prime}}_{\frac{d}{dt}\left(\sum m_i \vec{r}_i^{\,\prime}\right) = 0} + \sum \frac{1}{2} m_i \dot{\vec{r}}_i^{\,\prime 2}$$

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + T'$$

Total Kinetic energy of a system equals the sum of the kinetic energy of ~~[illegible]~~ the total mass moving with velocity of the center of mass and the kinetic energy in the CM frame

$$\vec{F}_i = \vec{F}_i^{\,ext} + \sum_{\substack{j \\ (j \neq i)}} \vec{F}_{ij}$$

Assume conservative forces

$$\vec{\nabla}_i = \frac{\partial}{\partial \vec{r}_i}$$

$$\vec{F}_i^{\,ext} = -\vec{\nabla}_i U_i^{ext}(\vec{r}_i)$$

$$\vec{F}_{ij} = -\vec{\nabla}_i U_{ij}(\vec{r}_i - \vec{r}_j)$$

Total potential energy?

$\vec{F}_1^{ext}$ $\vec{F}_{12}$

$\vec{F}_{21}$ $\vec{F}_2^{ext}$

Consider 2 particles.

$$U \stackrel{?}{=} U_1^{ext} + U_2^{ext} + \cancel{U_{12} + U_{21}}$$

No!

(4 forces $\Rightarrow$ 4 potentials?) No

Interparticle potential energy is not associated with either particle but rather with the pair.] emphasize

[eg wrong to think of grav potential en of earth, rather joint potential en of earth-sun system]

$$U = U_1^{ext} + U_2^{ext} + U_{12}(\vec{r}_1 - \vec{r}_2)$$
$$\qquad\qquad\qquad\qquad \text{''}$$
$$\qquad\qquad\qquad\qquad U_{21}(\vec{r}_1 - \vec{r}_2)$$

[Why? U_{12} is work done in bringing two particles together.] Bring first in, no work, ~~...~~

$\longrightarrow$ 1 $\qquad W_1 = 0$

1 $\quad$ 2 $\longleftarrow$ $\qquad W_2 = \int \vec{F}_{21} \cdot d\vec{r}$

2 $\longleftarrow$ $\qquad W_2 = 0$

$\longrightarrow$ 1 $\quad$ 2 $\qquad W_1 = \int \vec{F}_{12} \cdot d\vec{r}$

[[diff paths $\quad F_{21}\,(-dr)$, $\quad F_{12}\,(dr) = -F_{21}(dr)$ } equal]]

~~[to calculate these forces for one particle at a time, really]~~

$$\cancel{U^{total} = \underbrace{\sum_i U_i^{ext}(\vec{r}_i)}_{U^{ext}} + \underbrace{\sum_{\substack{\text{pairs}\\ \text{of particles}}} U_{ij}(\vec{r}_i - \vec{r}_j)}_{U^{int}}}$$

2 particles

[Do this w/ just 1 + 2 first]

$U_{12}(\vec{r}_i - \vec{r}_j)$ yields both $\vec{F}_{12}$ and $\vec{F}_{21}$, ~~and $\vec{F}_{ij} = -\vec{F}_{ji}$~~

Proof $\vec{F}_{12} = -\vec{\nabla}_1 U_{12}(\vec{r}_{12}) = -\frac{\partial}{\partial \vec{r}_i} U_{ij}(\vec{r}_{ij}) = -\frac{\partial}{\partial \vec{r}_{ij}} U_{ij}(\vec{r}_{ij})$

$$\boxed{\text{do } \frac{\partial}{\partial x} f(x-y) = \frac{\partial f(z)}{\partial z} \underbrace{\frac{dz}{dx}}_{1} = \frac{\partial f}{\partial z}}$$

$$\frac{\partial \vec{r}_{ij}}{\partial \vec{r}_i} = \mathbb{1} \qquad \frac{\partial \vec{r}_{ij}}{\partial \vec{r}_j} = -\mathbb{1}$$

$$\vec{F}_{ji} = -\vec{\nabla}_j U_{ji} = -\vec{\nabla}_j U_{ij} =$$

$$= -\frac{\partial}{\partial \vec{r}_j} U_{ij}(\vec{r}_{ij}) = +\frac{\partial}{\partial \vec{r}_{ij}} U_{ij}(\vec{r}_{ij}) = -\vec{F}_{ij}$$

(3rd law ✓)

Do this w/ 1 + 2 completely

$$U^{total} = \sum U_i^{ext}(\vec{r}_i) + \sum_{pairs} U_{ij}(\vec{r}_i - \vec{r}_j)$$

$$\sum_{pairs} U_{ij}(\vec{r}_i - \vec{r}_j) = \sum_{\substack{i,j\\ i<j}} U_{ij} \quad \text{internal} = \frac{1}{2} \sum_{\substack{i,j\\ i \neq j}} U_{ij}$$

Gravity:

$$U_{int} = \sum_{\substack{i,j\\ i<j}} \frac{G m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

Conservation of energy of a conservative system

$f(\vec r)$

$$\frac{df(\vec r)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$= (\vec\nabla f)\cdot\dot{\vec r}$$

$$E = T + U$$

$$= \sum \tfrac{1}{2} m \dot{\vec r}_i^{\,2} + \sum U_i^{ext}(\vec r_i) + \underbrace{\sum_{\substack{i,j\\ i<j}} U_{ij}(\vec r_i - \vec r_j)}_{\frac{1}{2}\sum_{\substack{i,j\\ (i\neq j)}} U_{ij}(\vec r_i - \vec r_j)}$$

$$\frac{dE}{dt} = \sum_i m_i \dot{\vec r}_i\cdot\ddot{\vec r}_i + \sum \vec\nabla_i U_i^{ext}\cdot\dot{\vec r}_i$$

$$+ \sum_{i<j}\left(\nabla_i U_{ij}\cdot\dot{\vec r}_i + \nabla_j U_{ij}\cdot\dot{\vec r}_j\right)$$

$$= \sum_i \Big(\overset{m_i\ddot{\vec r}_i}{\cancel{\vec F_i^{ext} + \sum_{\substack{j\\(j\neq i)}} \vec F_{ij}}}\Big)\cdot\dot{\vec r}_i + \sum\left(-\vec F_i^{ext}\cdot\dot{\vec r}_i\right)$$

$$+ \underbrace{\frac{1}{2}\sum_{i\neq j}\left(-\vec F_{ij}\cdot\dot{\vec r}_i - \vec F_{ji}\cdot\dot{\vec r}_j\right)}_{\sum_{\substack{i,j\\ i\neq j}}\left(-\vec F_{ij}\cdot\dot{\vec r}_i\right)}$$

$$= \sum_i \underbrace{\Big(m_i\ddot{\vec r}_i - \vec F_i^{ext} - \sum_{j\neq i} F_{ij}\Big)}_{0}\cdot\dot{\vec r}_i = 0$$

Consider a gravitating system of masses

Potential energy of test mass

$$U_{test} = -\sum_i \frac{G m m_i}{|\vec r - \vec r_i|}$$

= work done to bring test mass from ∞ to $\vec r$.

$$= -\int \vec F \cdot d\vec r = -\int \sum_i \vec F_i \cdot d\vec r = \sum_i \left(-\int \vec F_i \cdot d\vec r\right) = \sum_i U_i$$

Gravitational potential $\equiv \dfrac{\text{Potential energy}}{\text{mass}}$

$$\Phi(\vec r) = \frac{U}{m} = -\sum \frac{G m_i}{|\vec r - \vec r_i|}$$

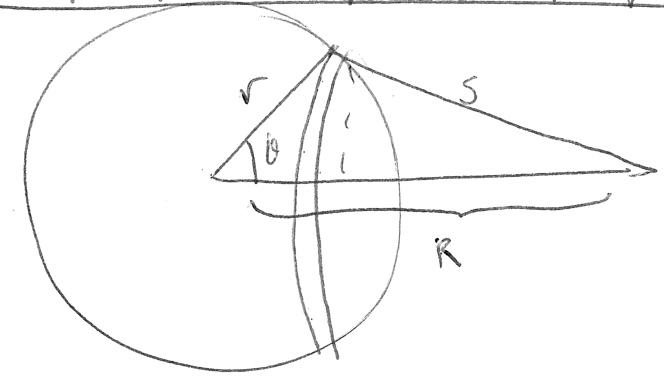
(doesn't depend on mass, only on position)

Continuous system

$$\Phi(\vec r) = -\int \frac{G \rho(\vec r')\, d^3\vec r'}{|\vec r - \vec r'|}$$

= Work ~~done on~~ done per unit mass to bring a ~~mass~~ particle from ∞ to $\vec r$

Potential of spherical shell of radius r



$$\Phi = -\int \frac{G\,\rho\, dV}{s} = -\int \frac{G\rho\, 2\pi r^2 dr \sin\theta\, d\theta}{s}$$

$$= -\frac{G}{2}\int \frac{\sin\theta\, d\theta}{s}$$

$$\rho = \frac{dM}{4\pi r^2 dr}$$

$$s^2 = (R - r\cos\theta)^2 + (r\sin\theta)^2$$

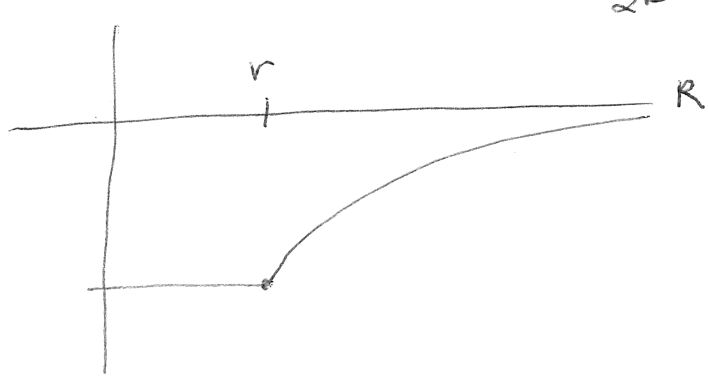
$$= R^2 - 2Rr\cos\theta + r^2$$

$$\cancel{2}s\, ds = +\cancel{2}Rr\sin\theta\, d\theta$$

$$\Phi = -\frac{G\,dM}{2Rr}\int ds$$

Outside shell: $\Phi = -\frac{G\,dm}{2Rr}\underbrace{\int_{R-r}^{R+r} ds}_{2r} = -\frac{G\,dM}{R} \Rightarrow |F| = \frac{GMm}{R^2}$

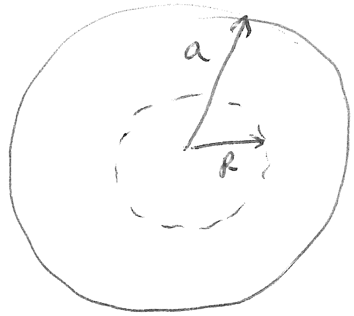
Inside shell $\Phi = -\frac{G\,dM}{2Rr}\underbrace{\int_{r-R}^{r+R} ds}_{2R} = -\frac{G\,dM}{r} \Rightarrow \vec{F} = 0$



Potential of sphere of radius a

Outside sphere $\quad \Phi = -\dfrac{GM}{R}$

Inside.



$$\Phi = -\int_0^R \frac{G\,dM}{R} - \int_R^a \frac{G\,dM}{r}$$

$$dM = 4\pi\rho r^2 dr$$

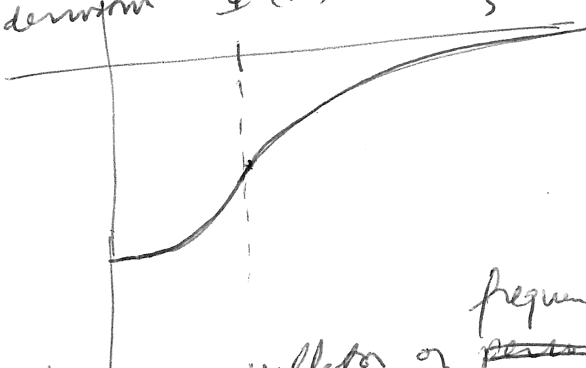
$$\cancel{\Phi = -\frac{4\pi G\rho}{R}\int_0^R r^2 dr}$$

$$\Phi = -\frac{4\pi G\rho R^2}{3} - 2\pi G\rho\,(a^2 - R^2)$$

$$= -2\pi G\rho\, a^2 + \frac{2\pi G\rho}{3} R^2$$

Continuity at $R = a$: $\quad \Phi(a) = -\frac{4\pi}{3} G\rho a^2 = -\frac{GM}{a}$ ✓

Continuity of derivative $\quad \Phi'(a) = \frac{4\pi G\rho a}{3} = \frac{GM}{a^2}$



$$U = m\Phi = \frac{1}{2}\underbrace{\left(\frac{4\pi}{3}G\rho m\right)}_{k} R^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

Inside sphere: harmonic oscillator of ~~period~~ frequency $\sqrt{\frac{4\pi}{3}G\rho}$

Same as orbital frequency!

earth: 90 mins

Calculus of variations

[Note: See my S' 2006 Relativity Notes]

function $y(x)$ ((machine that takes a #, gives back a #))

functional $J[y(x)]$ = function of a function

((takes a fcn, gives a #))

Ex.

1. $A[y(x)] = \frac{1}{L}\int_0^L y(x)\,dx$ ((averaging))

2. $\langle x \rangle = X[\psi(x)] = \int \psi^*(x)\, x\, \psi(x)\, dx$

3. $L[y(x)] = \int ds = \int \sqrt{dx^2+dy^2}$ ((arc length of a curve))

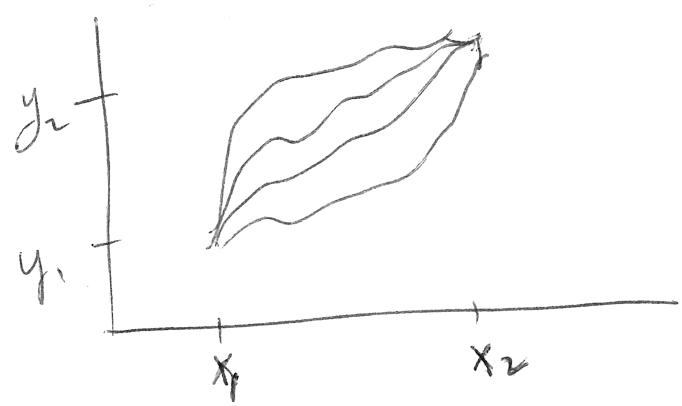
$$= \int_{x_1}^{x_2} dx \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$J[y(x)] = \int_{x_1}^{x_2} dx\, f(\,y(x),\, y'(x);\, x\,)$$

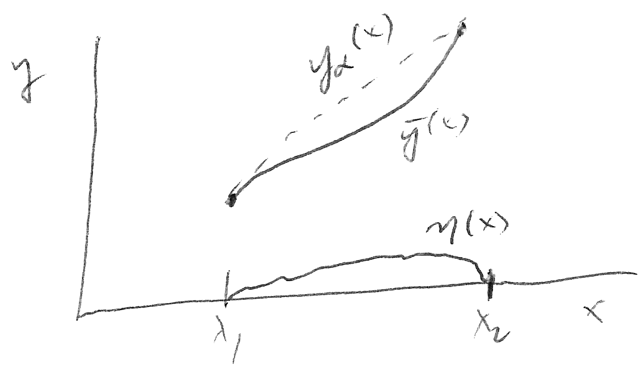
[May want to minimize J,
eg find shortest path btw 2 pts]

Which function $y(x)$ with $y(x_1)=y_1$ and $y(x_2)=y_2$
minimizes $J[y(x)]$?

~~with b.c. $y(x_1)=y_1$~~
~~$y(x_2)=y_2$~~



~~Denote~~ Let $\bar{y}(x)$ minimize $J[y(x)]$, $\bar{y}(x_1)=y_1$
$\bar{y}(x_2)=y_2$

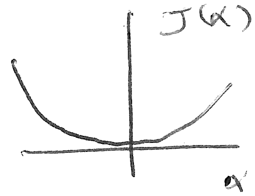


Let $\eta(x)$ denote an arbitrary function s.t. $\eta(x_1)=0$
$\eta(x_2)=0$

Define $y_\alpha(x) = \bar{y}(x) + \alpha\eta(x)$

$y_\alpha(x)$ satisfies b.c. $y_\alpha(x_1)=y_1$, $y_\alpha(x_2)=y_2$

$J[y_\alpha(x)] \geq J[\bar{y}(x)]$



[Want to go slow
expand on various pts]

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha} = 0$$

Instead, I started by Taylor expanding f then took the integral

$$J(\alpha) = J[y_\alpha(x)] = \int_{x_1}^{x_2} dx\, f(\underbrace{y_\alpha}_{y+\alpha\eta},\ \underbrace{y'_\alpha}_{y'+\alpha\eta'},\ x)$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y}(y_\alpha) \underbrace{\frac{dy_\alpha}{d\alpha}}_{\eta} + \frac{\partial f}{\partial y'}(y_\alpha) \underbrace{\frac{dy'_\alpha}{d\alpha}}_{\eta'} \right]$$

$$0 = \left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y}(\bar{y})\,\eta + \underbrace{\frac{\partial f}{\partial y'}(\bar{y})\,\eta'}_{\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta\right] - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta} \right]$$

$$= \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta + \underbrace{\left.\left(\frac{\partial f}{\partial y'}\right)\eta\,\right|_{x_1}^{x_2}}_{\frac{\partial f}{\partial y'}(x_2)\,\eta(x_2) - \frac{\partial f}{\partial y'}(x_1)\,\eta(x_1)}$$

$$\eta(x_2) = 0, \quad \eta(x_1) = 0$$

Since this vanishes for any $\eta(x)$, integrand itself vanishes

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Euler's eqn

Shortest distance between 2 ~~pts in str~~ pts is a straight line

$$L[y(x)] = \int dx \underbrace{\sqrt{1+(y')^2}}_{f}$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

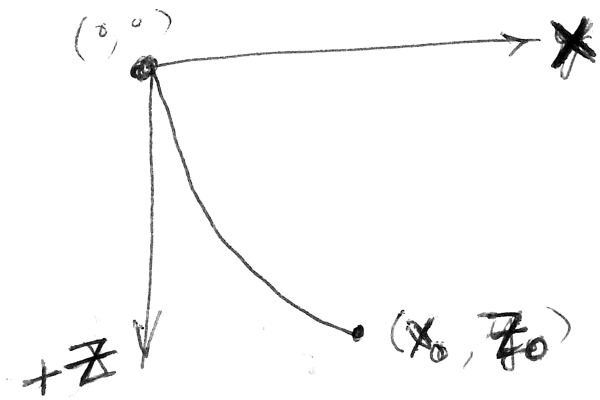
$$\frac{y'}{\sqrt{1+y'^2}} = c$$

$$y'^2 = c^2 + c^2 y'^2$$

$$y' = \frac{c}{\sqrt{1-c^2}} = D$$

$$y = Dx + E \quad \checkmark$$

Brachistrochrone: path of least time



$$T = \int dt = \int \frac{ds}{v}$$

$$ds^2 = dx^2 + dy^2 = dx^2\left(1 + \left[\frac{dz}{dx}\right]^2\right) = dz^2\left(1 + \left[\frac{dx}{dz}\right]^2\right)$$

$$ds = dz\sqrt{1 + \left(\frac{dx}{dz}\right)^2}$$

$$E = T + U = \frac{1}{2}mv^2 - mgz = 0$$

$$v = \sqrt{2gz}$$

$$T = \int dz \sqrt{\frac{1 + \left(\frac{dx}{dz}\right)^2}{2gz}}$$

$$T\left[x(z), z\right]$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial\left(\frac{dx}{dz}\right)} = \cancel{\tfrac{1}{2}}\,\frac{\cancel{2}\left(\frac{dx}{dz}\right)}{\sqrt{1+\left(\frac{dx}{dz}\right)^2}\,\sqrt{2gz}}$$

$$\frac{d}{dz}\left(\frac{\partial T}{\partial\left(\frac{dx}{dz}\right)}\right) - \frac{\partial T}{\partial x} = 0$$

$$\frac{d}{dz}\left\{\frac{\frac{dx}{dz}}{\sqrt{1+\left(\frac{dx}{dz}\right)^2}\,\sqrt{2gz}}\right\} = 0$$

$$\frac{dx}{dz} = \sqrt{K}\,\sqrt{1+\left(\frac{dx}{dz}\right)^2}\,\sqrt{2gz}$$

$$\left(\frac{dx}{dz}\right)^2 = \cancel{K}\,2gKz\left(1+\left(\frac{dx}{dz}\right)^2\right)$$

$$\left(\frac{dx}{dz}\right)^2\left(1-2gKz\right) = 2gKz$$

$$\frac{dx}{dz} = \sqrt{\frac{2gKz}{1-2gKz}}$$

$$x = \int\sqrt{\frac{2gKz}{1-2gKz}}$$

H7

$$z = \frac{1}{4gK}(1-\cos\theta)$$

$$dz = \frac{\sin\theta\, d\theta}{4gK}$$

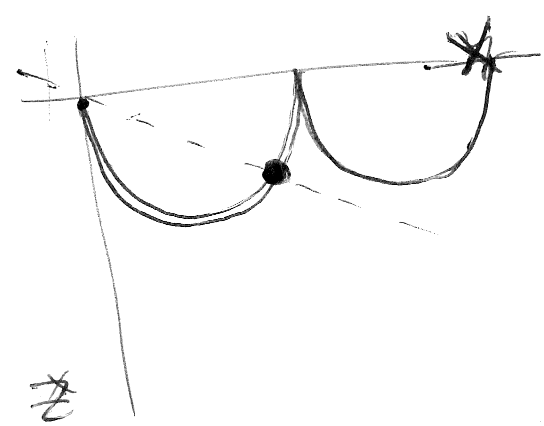
$$x = \int \sqrt{\frac{\frac{1}{2}(1-\cos\theta)}{1-\frac{1}{2}(1-\cos\theta)}}\ \frac{\sin\theta\, d\theta}{4gK}$$

$$= \int \sqrt{\frac{\cancel{\tfrac{1}{2}}(1-\cos\theta)}{\cancel{\tfrac{1}{2}}(1+\cos\theta)}}\ \frac{\sqrt{1-\cos^2\theta}}{4gK}\, d\theta$$

$$= \frac{1}{4gK}\int (1-\cos\theta)\, d\theta$$

$$x = \frac{1}{4gK}(\theta - \sin\theta) + K'$$

$x=0,\ z=0 \Rightarrow K'=0$



$$x = \frac{1}{4gK}(\theta - \sin\theta)$$

$$z = \frac{1}{4gK}(1-\cos\theta)$$

$$x_0 = \frac{1}{4gK}(\theta_0 - \sin\theta_0)$$

$$y_0 = \frac{1}{4gK}(1-\cos\theta_0)$$

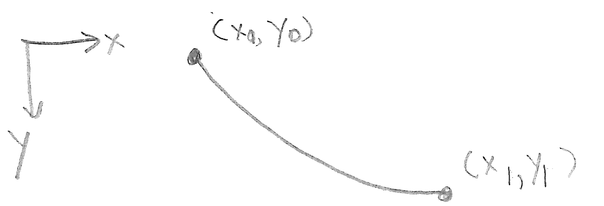
$$\frac{y_0}{x_0} = \frac{1-\cos\theta_0}{\theta_0 - \sin\theta_0} \Rightarrow \text{solve for } \theta_0, \text{ then } 4gK = \frac{\theta_0 - \sin\theta_0}{x_0}$$ ✓

June 2006 at Strings conf.

- Johann Bernoulli posed this as a challenge 1696
- Newton solved this in 1 day in 1696
  Leibniz, L'Hopital & Bernoullis also found it.
- Huygens ~~also~~ noted that cycloid is a tautochrone.
  (1673)

1

Brachistochrone



$$\left[\begin{array}{r} \frac{1}{2}mv^2 - mgy = \frac{1}{2}mv_0^2 - mgy_0 \\ v = \sqrt{v_0^2 + 2g(y-y_0)} \\ \text{Let } v_0 = 0 \text{ and } y_0 = 0 \Rightarrow v = \sqrt{2gy} \end{array}\right]$$

Time of trajectory $T[y(x)] = \int \frac{ds}{v} = \int \frac{\sqrt{1+(y')^2}\,dx}{\sqrt{2gy}}$

$$\Rightarrow L = \sqrt{\frac{1+y'^2}{y}}, \quad \frac{\partial L}{\partial y} = -\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}}, \quad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{y(1+y'^2)}}$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{y''}{\sqrt{y(1+y'^2)}} - \frac{1}{2}\frac{y'^2}{\sqrt{y^3(1+y'^2)}} - \frac{y'^2 y''}{\sqrt{y(1+y'^2)^3}}$$

$$= \frac{y(1+y'^2)y'' - \frac{1}{2}y'^2(1+y'^2) - yy'^2y''}{\sqrt{y^3(1+y'^2)^3}}$$

$$= \frac{yy'' - \frac{1}{2}y'^2(1+y'^2)}{\sqrt{y^3(1+y'^2)^3}} = -\frac{1}{2}\frac{(1+y'^2)^2}{\sqrt{y^3(1+y'^2)}}$$

$$2yy'' - y'^2(1+y'^2) = -(1+y'^2)^2$$

$$\Rightarrow \boxed{2yy'' + 1 + y'^2 = 0}$$

Better to parametrize curve: $x(p), y(p)$

Then $y' = \frac{\dot{y}}{\dot{x}}, \quad y'' = \frac{\ddot{y}}{\dot{x}^2} - \frac{\dot{y}\ddot{x}}{\dot{x}^3} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$

$$\Rightarrow \boxed{2y(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \dot{x}^3 + \dot{x}\dot{y}^2 = 0}$$

since $\frac{\partial L}{\partial x} = 0$

one can use

Beltrami identity

$$L - y'\frac{\partial L}{\partial y'} = C$$

$$\sqrt{\frac{1+y'^2}{y}} - \frac{y'^2}{\sqrt{y(1+y'^2)}} = C$$

$$\frac{1}{\sqrt{y(1+y'^2)}} = C$$

$$y(1+y'^2) = k^2$$

$$\Rightarrow \begin{cases} x = \frac{k^2}{2}(p - \sin p) \\ y = \frac{k^2}{2}(1 - \cos p) \end{cases}$$

Brachistochrone (cont)

$$T = \int \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{2gy}}\, dp$$

$$L = \sqrt{\frac{\dot{x}^2+\dot{y}^2}{y}}, \quad \frac{\partial L}{\partial y} = -\frac{1}{2}\sqrt{\frac{\dot{x}^2+\dot{y}^2}{y^3}}, \quad \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}$$

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}$$

$$\frac{d}{dp}\left(\frac{\dot{x}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}\right) = 0 \Rightarrow \boxed{\dot{x}^2 = \frac{1}{2}K\, y(\dot{x}^2+\dot{y}^2)}$$

$$\hookrightarrow \frac{\ddot{x}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}} + \dot{x}\frac{d}{dp}\left(\frac{1}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}\right) = 0$$

$$\frac{d}{dp}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\ddot{y}}{\sqrt{y(\dot{x}^2+\dot{y}^2)}} + \dot{y}\underbrace{\frac{d}{dp}\left(\frac{1}{\sqrt{y(\dot{x}^2+\dot{y}^2)}}\right)}_{-\frac{\ddot{x}}{\dot{x}\sqrt{y(\dot{x}^2+\dot{y}^2)}}} = -\frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{\sqrt{y^3(\dot{x}^2+\dot{y}^2)}}$$

$$\ddot{y} - \frac{\dot{y}}{\dot{x}}\ddot{x} = -\frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y}$$

$$\boxed{2y(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \dot{x}(\dot{x}^2+\dot{y}^2) = 0}$$ as before

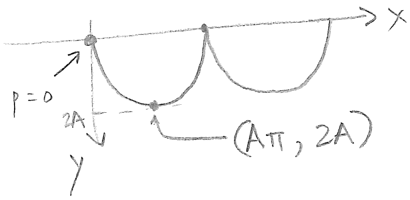
Brachistochrome (cont)

To show: cycloid = brachistrochrom

$$\left.\begin{aligned} \dot{x}^2 &= \tfrac{1}{2} K y (\dot{x}^2 + \dot{y}^2) \\ 2y(\dot{x}\ddot{y} - \dot{y}\ddot{x}) &+ \dot{x}(\dot{x}^2 + \dot{y}^2) = 0 \end{aligned}\right\} \quad K y^2 (\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \dot{x}^3 = 0$$

Let $\begin{cases} x = A(p - \sin p) \\ y = A(1 - \cos p) \end{cases} \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = A \sin p \end{cases} \Rightarrow \dot{x}^2 + \dot{y}^2 = 2A^2(1 - \cos p) = 2Ay$

$\downarrow$



Check $\dot{x}^2 = \frac{1}{2} K y (\dot{x}^2 + \dot{y}^2)$

$y^2 = \frac{1}{2} K y (2Ay) \Rightarrow K = \frac{1}{A}$

Check $K y^2 (\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \dot{x}^3 = 0$

$$\frac{1}{A} y^2 \underbrace{\left(\underbrace{A[1-\cos p][A \cos p] - [A \sin p]^2}_{A^2 \cos p - A^2}\right)}_{-Ay} + y^3 = 0 \checkmark$$

Hence $\frac{y_1}{x_1} = \left(\frac{1 - \cos p_1}{p_1 - \sin p_1}\right)$ determines p_1 $\longrightarrow$ so if $\frac{y_1}{x_1} \gtrless \frac{2}{A}$ then $p_1 \lessgtr \pi$

ie  vs 

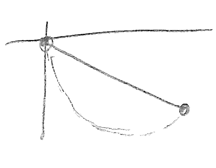
then $y_1 = A(1 - \cos p_1)$ determines A

Then $T(x_1, y_1) = \int \sqrt{\frac{\dot{x}^2 + \dot{y}^2}{2gy}}\, dp = \int_0^{p_1} \sqrt{\frac{A}{g}}\, dp = \sqrt{\frac{A}{g}}\, p_1$

along brachistochrome

(eg if $x_1 \to 0$, then $p_1 \to 0$ so $y_1 \to A \frac{1}{2} p_1^2$ so $T(0, y_1) = \sqrt{\frac{2y_1}{g}}$ ✓)

Consider straight line path $\Rightarrow \begin{cases} x = x_1 p \\ y = y_1 p \end{cases} \Rightarrow T(x_1, y_1) = \sqrt{\frac{x_1^2 + y_1^2}{2gy_1}} \int_0^1 \sqrt{p}\, dp = \sqrt{\frac{2(x_1^2 + y_1^2)}{9y_1}}$



Also: $T = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2s}{g \sin\theta}} = \sqrt{\frac{2\sqrt{x_1^2 + y_1^2}}{g\left(\frac{y_1}{\sqrt{x_1^2 + y_1^2}}\right)}} = \sqrt{\frac{2(x_1^2 + y_1^2)}{gy_1}}$ ✓

plot this.

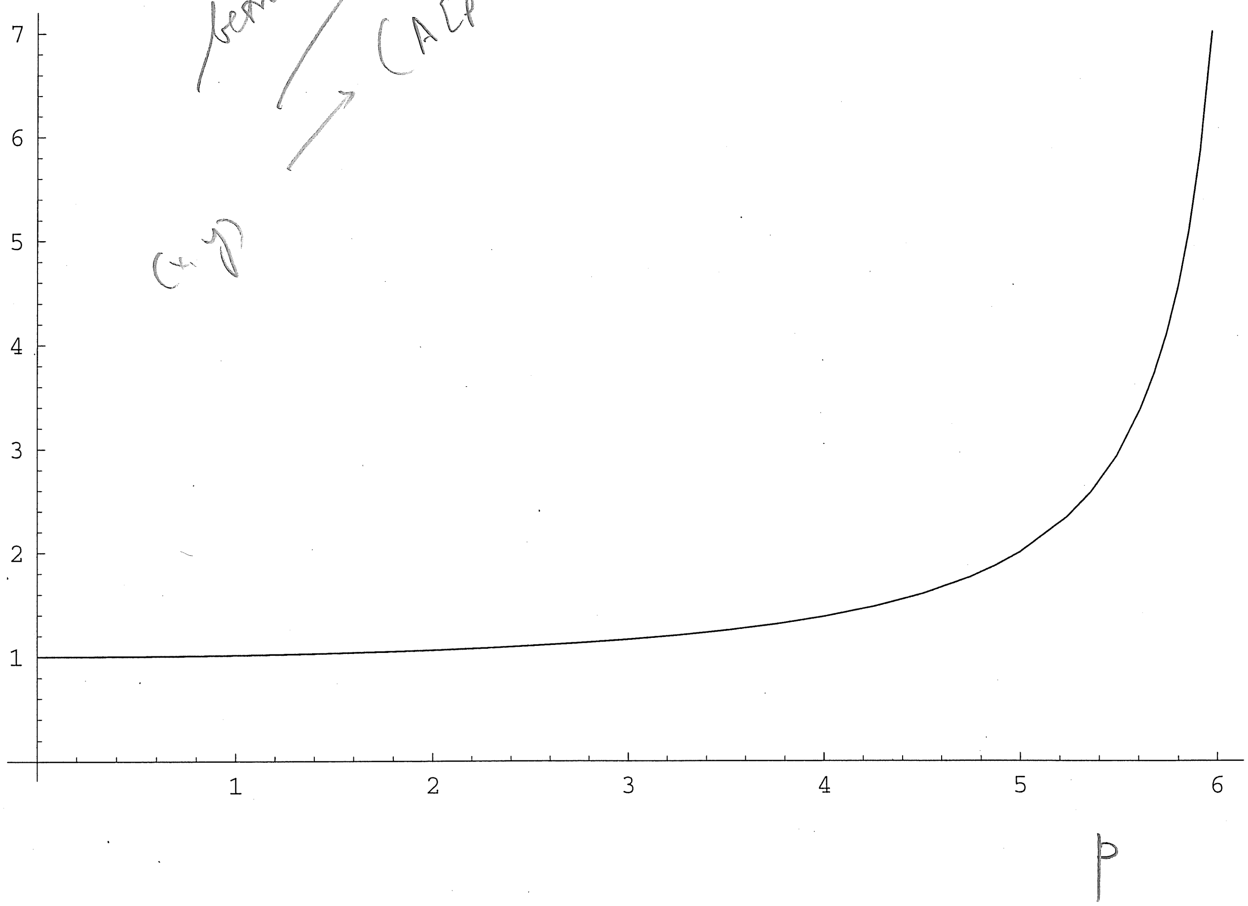
Compare times: $\frac{T_{\text{straight}}}{T_{\text{cyc}}} = \frac{\sqrt{\frac{2(x_1^2 + y_1^2)}{gy_1}}}{\sqrt{\frac{A}{g} p_1^2}} = \sqrt{\frac{2[(p - \sin p)^2 + (1 - \cos p)^2]}{p_1^2 [1 - \cos p]}} = \sqrt{\frac{2(p_1^2 - 2p_1 \sin p_1 + 2(1 - \cos p_1))}{p_1^2 (1 - \cos p_1)}}$

$(p_1 \to 0 \Rightarrow \text{ratio} \to 1)$

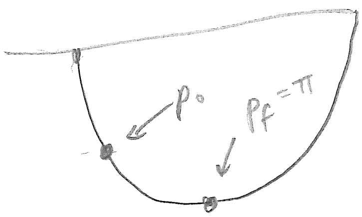
ratio of time between a straight line path + a brachistrone

between $(0,0)$ and $(A[p-\sin p], -A(1-\cos p))$

(x,y)



Huygens noticed (1673) that a cycloid is ~~[illegible]~~ a <u>tautochrone</u>



$$x = A(p - \sin p)$$
$$y = A(1 - \cos p)$$

Start at p_0, end at $p_f = \pi$

$$T = \int_{p_0}^{p_f} \sqrt{\frac{\dot{x}^2 + \dot{y}^2}{2g(y - y_0)}} = \sqrt{\frac{A}{g}} \int_{p_0}^{p_f} \sqrt{\frac{1 - \cos p}{\cos p_0 - \cos p}}\, dp$$

As calc'd before, $p_0 = 0 \Rightarrow T = \sqrt{\frac{A}{g}}\,\pi$

But now (help from MathWorld)

$$\cos p = 2\cos^2\left(\tfrac{p}{2}\right) - 1$$
$$1 - \cos p = 2\sin^2\left(\tfrac{p}{2}\right)$$

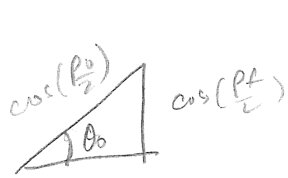
$$\Rightarrow \int_{p_0}^{p_f} \sqrt{\frac{1 - \cos p}{\cos p_0 - \cos p}}\, dp = \int_{p_0}^{p_f} \frac{\sin\left(\frac{p}{2}\right) dp}{\sqrt{\cos^2\left(\frac{p_0}{2}\right) - \cos^2\left(\frac{p}{2}\right)}}$$

$u = \dfrac{\cos\left(\frac{p}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}$ ~~$u = \dots$~~

$$= \int_{p_0}^{p_f} \frac{\frac{\sin\left(\frac{p}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}\, dp}{\sqrt{1 - \frac{\cos^2\left(\frac{p}{2}\right)}{\cos^2\left(\frac{p_0}{2}\right)}}} = \int_1^{\frac{\cos\left(\frac{p_f}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}} \frac{-2\, du}{\sqrt{1 - u^2}}$$

$du = -\dfrac{\frac{1}{2}\sin\left(\frac{p}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}$

$$= 2 \arcsin u \Big|_{u = \frac{\cos\left(\frac{p_f}{2}\right)}{\cos\left(\frac{p_0}{2}\right)}}^{\text{"}1\text{"}}$$



$$= \pi - \theta_0 \xrightarrow[p_f \to \pi]{} \pi, \text{ indep of } p_0$$

$\cos\left(\frac{p_f}{2}\right) \to 0$

$\theta_0 \to 0$

Thus time from p_0 to π is <u>indep of starting pt.</u>

~~(a pendulum on a circle differs)~~ $\boxed{\therefore \text{ amplitude-indep pendulum}}$