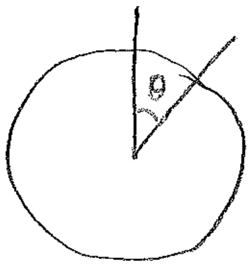
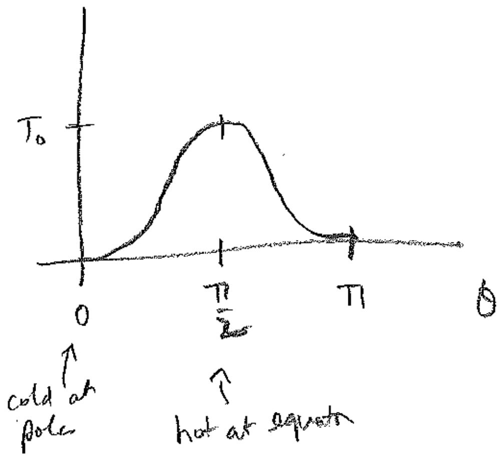
Calculate steady-state temp. distribution ($\Rightarrow \nabla^2 T = 0$)

of a solid sphere of radius R

whose surface temperature is held at $T(R,\theta) = T_0 \sin^2\theta$





B.C. are azimuthally symmetric (indep of ϕ) so

$$T(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$$

Temp should be finite at center of sphere $\Rightarrow B_\ell = 0$

$$T(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = T_0 \sin^2\theta$$
$$= T_0(1-\cos^2\theta)$$

$$\sum A_\ell R^\ell P_\ell(x) = \underbrace{T_0 - T_0 x^2}$$

even, 2nd order polynomial

Is it $P_2(x) = \frac{1}{2}(3x^2-1)$

Also need $P_0(x) = 1$

$$A_0 \underbrace{P_0(x)}_{1} + A_2 R^2 \underbrace{P_2(x)}_{\frac{1}{2}(3x^2-1)} = T_0 - T_0 x^2$$

Match coefficients

x^0: $$A_0 - \frac{1}{2} A_2 R^2 = T_0$$

x^2: $$\frac{3}{2} A_2 R^2 = -T_0$$

$$A_2 = -\frac{2T_0}{3R^2}$$

$$A_0 = T_0 + \frac{1}{2} R^2 \left(-\frac{2T_0}{3R^2}\right) = \frac{2}{3} T_0$$

$$\Rightarrow \boxed{T(r,\theta) = \frac{2}{3} T_0 \; P_0(\cos\theta) - \frac{2T_0}{3} \left(\frac{r}{R}\right)^2 P_2(\cos\theta)}$$

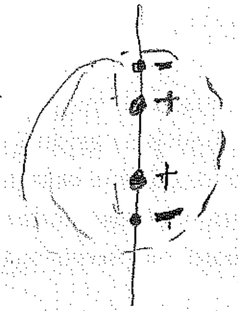
check the b.c. $T(R,\theta) = \frac{2}{3} T_0 - \frac{2T_0}{3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

$= T_0 - T_0 \cos^2\theta$ ✓

At center of sphere $T(0,\theta) = \underline{\underline{\frac{2}{3} T_0}}$

Consider an ~~azimuthally symmetric~~ charge distribution near the origin. $\nabla^2 V = -\frac{\rho}{\epsilon_0}$.

e.g.



In the region exterior to the charge distribution

$$\nabla^2 V = 0 \quad \text{(charge density vanishes)}$$

$$V = \sum \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$A_\ell = 0$ for $\ell > 0$ because $V \not\to \infty$ as $r \to \infty$

$$V = \left(A_0 + \frac{B_0}{r}\right) P_0 + \frac{B_1}{r^2} P_1 + \frac{B_2}{r^3} P_2 + \frac{B_3}{r^4} P_3 + \dots$$

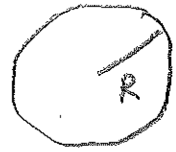
$$= A_0 + \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \dots$$

prop to total charge

prop to quadrupole moment

prop to dipole moment

Spherical turkey (sous vide)



A turkey, initially at temp $T_{init} = 40\,F$
is immersed in a bath at temp $T_{bath} = 212F$ at $t = 0$
How long until center of turkey reaches $T_{safe} = 165\,F$?

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

$$\tilde{T} = T_{bath} - T \quad \Rightarrow \quad \frac{\partial \tilde{T}}{\partial t} = D\nabla^2 \tilde{T}$$

Boundary conditions + initial conditions have spherical symmetry

$$\Rightarrow \tilde{T} = \tilde{T}(r, t)$$

Seperable solution $\tilde{T}(r,t) = f(r)\,g(t)$

$$\underbrace{\frac{1}{Dg}\frac{dg}{dt}}_{-k^2} = \frac{1}{f}\nabla^2 f = \underbrace{\frac{1}{fr^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right)}_{-k^2}$$

$$\hookrightarrow g = g_0 e^{-Dk^2 t}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = -k^2 f$$

check! $\longrightarrow$ $\frac{1}{r}\frac{d^2}{dr^2}(rf)$

$$\frac{d^2}{dr^2}(rf) + k^2(rf) = 0$$

$$\Rightarrow rf = A\sin(kr) + B\cos(kr)$$

$$f = A\frac{\sin(kr)}{r} + B\underbrace{\frac{\cos(kr)}{r}}_{\to\infty \text{ as } r\to 0}$$

so $B = 0$

$$f = A\,\frac{\sin(kr)}{r}$$

Boundary conditions $\tilde{T}(R,t) = 0 \;\Rightarrow\; \sin(kR) = 0, \quad k_n = \frac{n\pi}{R}$

$$\tilde{T}(r,t) = \sum A_n \frac{\sin(k_n r)}{r} e^{-Dk_n^2 t}$$

Initial conditions $\tilde{T}(r,0) = T_{bath} - T_{init} \equiv \Delta T$

$$\sum_n A_n \frac{\sin(k_n r)}{r} = \Delta T$$

$$\sum A_n \sin(k_n r) = r\,\Delta T$$

$$\underbrace{\sum A_n \underbrace{\int_0^R \sin(k_n r)\,\sin(k_m r)\,dr}_{\delta_{nm}\frac{R}{2}}}_{\frac{A_m R}{2}} = \Delta T \int_0^R r \sin(k_m r)\,dr$$

$u = r \quad dv = \sin(k_m r)\,dr$

$du = dr \quad v = -\frac{1}{k_m}\cos(k_m r)$

$$= \Delta T\left[-\frac{r}{k_m}\cos(k_m r)\Big|_0^R + \frac{1}{k_m}\underbrace{\int_0^R \cos(k_m r)\,dr}_{\frac{\sin(k_m r)}{k_m}\Big|_0^R}\right]$$

$$= -\Delta T\,\frac{R^2}{\pi m}\underbrace{\cos(k_m R)}_{(-1)^m}$$

$$A_m = -\frac{2\,\Delta T\,R}{\pi m}(-1)^m$$

$$\tilde{T}(r,t) = -2\,\Delta T\cdot\frac{R}{r}\sum_{m=1}^{\infty}\frac{(-1)^m}{\pi m}\sin\left(\frac{\pi m r}{R}\right) e^{-\frac{D\pi^2 n^2}{R^2}t}$$

Center of turkey: $\dfrac{\sin\left(\frac{\pi m r}{R}\right)}{\pi m} \longrightarrow \dfrac{r}{R}$

$$\tilde{T}(0,t) = -2\Delta T \sum (-1)^m e^{-\frac{D\pi^2 n^2 t}{R^2}}$$

[Note: $\tilde{T}(0,0) = -2\Delta T\,[\underbrace{-1+0-1+0-\cdots}_{-\frac{1}{2}}] = \Delta T$ ✓]

Keep 1st term

$$\tilde{T}(0,t) \approx 2\,\Delta T\, e^{-\frac{\pi^2 D t}{R^2}}$$

$$T_b - T_{safe} = 2\,(T_b - T_{init})\, e^{-\frac{\pi^2 D t_{safe}}{R^2}}$$

$$t_{safe} = \frac{R^2}{\pi^2 D} \ln\left(2\,\frac{T_b - T_{init}}{T_b - T_{safe}}\right)$$

$$V = \frac{4}{3}\pi R^3 \Rightarrow t_{safe} = \underbrace{\left(\frac{3}{4\pi}\right)^{2/3}}_{0.385} \frac{V^{2/3}}{\pi^2 D} \ln\left(2\,\frac{(T_b - T_{init})}{(T_b - T_{safe})}\right) = 4.3 \text{ hrs}$$

for 10 lb turkey

For a cubical turkey: $t_{safe} = \dfrac{1}{3}\dfrac{V^{2/3}}{\pi^2 D} \ln\left(\underbrace{\dfrac{64}{\pi^2}}_{2.064}\,\dfrac{T_b - T_{init}}{T_b - T_{safe}}\right) = 3.7 \text{ hrs}$

for 10 lb.

Spherical turkey takes a little longer
(cubical turkey has 25% more SA per volume)

why does it take less time to cook a cubical turkey than a spherical turkey of the same volume?

<u>Cube</u>

$$L = V^{1/3}$$

$$A = 6L^2 = 6V^{2/3}$$

<u>sphere</u>

$$R = \left(\frac{3V}{4\pi}\right)^{1/3}$$

$$A = 4\pi R^2 = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} V^{1/3}$$

$$= \left(\frac{(4\pi)^{3/2}\, 3}{4\pi}\right)^{2/3} V^{1/3}$$

$$= \left(\sqrt{4\pi}\; 3\right)^{2/3}$$

$$= \underbrace{\underbrace{(4\pi)^{1/3}}_{2.32}\ \underbrace{3^{2/3}}_{2.08}}_{4.836}\ V^{1/3}$$

Sphere has surface area 80.6% <u>less</u> than a cube of same volume

Cubical turkey (cons vide)

ⓐ A turkey, initially at temp. T_{init} is immersed in a bath at temp T_b at $t=$

$$\frac{\partial T}{\partial t} = D\nabla^2 T \Rightarrow \frac{\partial \tilde{T}}{\partial t} = D\nabla^2 \tilde{T} \quad \text{where} \quad \tilde{T} = T_b - T$$

Cubical turkey, w/ b.c. $\tilde{T}=0$ at $x=0, y=0, z=0, x=L, y=L, z=L$

$$\tilde{T} = \sum_{n_1 n_2 n_3} A_{n_1 n_2 n_3} \sin(k_{n_1} x) \sin(k_{n_2} y) \sin(k_{n_3} z)\, e^{-\alpha_{n_1 n_2 n_3} t}$$

$$k_{n_i} = \frac{\pi}{L} n_i, \quad \alpha_{n_1 n_2 n_3} = D(k_{n_1}^2 + k_{n_2}^2 + k_{n_3}^2) = \frac{D\pi^2}{L^2}(n_1^2 + n_2^2 + n_3^2)$$

Initial condition $\tilde{T} = T_b - T_{init} = \Delta T$

$$\Delta T = \sum A_{n_1 n_2 n_3} \sin(k_{n_1} x) \sin(k_{n_2} y) \sin(k_{n_3} z)$$

$$\Rightarrow A_{n_1 n_2 n_3} = \frac{64\,\Delta T}{\pi^3 n_1 n_2 n_3} \quad \text{for } n_1, n_2, n_3 \text{ odd}; \ 0, \text{ otherwise}$$

Keep just first term

$$\tilde{T} = \frac{64\,\Delta T}{\pi^3} \underbrace{\sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \sin\left(\frac{\pi z}{L}\right)}_{=1 \text{ at center}} e^{-\frac{3D\pi^2}{L^2} t}$$

$T = T_{safe}$ when $\tilde{T} = T_b - T_{safe} \Rightarrow \left(\frac{T_b - T_{safe}}{T_b - T_{init}}\right)\frac{\pi^2}{64} = e^{-\frac{3\pi^2 D}{L^2} t_{safe}}$

$$\boxed{t_{safe} = \frac{L^2}{3\pi^2 D} \ln\left(\frac{64}{\pi^2}\,\frac{T_b - T_{init}}{T_b - T_{safe}}\right)}$$

$D = 1.4 \times 10^{-7}\,m^2/s \Rightarrow \frac{L^2}{3\pi^2 D} = 6.568E4\,s = 1.82\,h$

ⓑ $10\,lbs \to 4.54\,kg = 4540\,cm^3 \Rightarrow L = 0.165\,m,$

$T_b = 212F,\ T_{init} = 40,\ T_{safe} = 165 \Rightarrow \frac{T_b - T_{init}}{T_b - T_{safe}} = 3.66,\ \ln\left(\frac{64}{\pi^3}(3.66)\right) = 2.02$

$$\boxed{t_{safe} = 3.7\ hrs}$$

ⓒ $t_{safe} \sim L^2 \sim V^{2/3}$