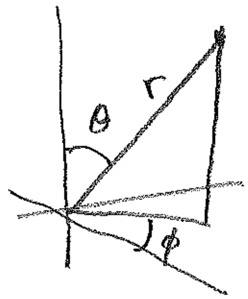
Laplace's eqn in spherical coordinates



$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

range:
$0 \le \phi < 2\pi$
$0 \le \theta \le \pi$

Try separable ansatz: $\Phi(r,\theta,\phi) = f(r)\, g(\theta)\, h(\phi)$

$$\frac{1}{r^2 f}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) + \frac{1}{r^2 \sin\theta\, g}\frac{d}{d\theta}\left(\sin\theta \frac{dg}{d\theta}\right) + \frac{1}{r^2 \sin^2\theta}\underbrace{\frac{1}{h}\frac{d^2 h}{d\phi^2}}_{\text{must be constant because rest of eqn is indep of } \phi} = 0$$

$h(\phi + 2\pi) = h(\phi)$

$$\Rightarrow \frac{1}{h}\frac{d^2 h}{d\phi^2} = -m^2$$

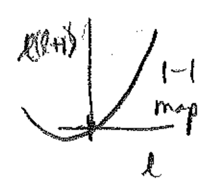
$$h = C_m \cos m\phi + D_m \sin m\phi$$

Multiply eqn by r^2:

$$\underbrace{\frac{1}{f}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right)}_{\text{does not depend on } \theta} + \underbrace{\frac{1}{\sin\theta\, g}\frac{d}{d\theta}\left(\sin\theta \frac{dg}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{\text{does not depend on } r} = 0$$

to obtain well-behaved solutions in θ
(ie finite at $\theta = 0$ and $\theta = \pi$)
this constant must be nonpositive

$-\ell(\ell+1)$



in fact, ℓ will have to be a non-negative integer

$\ell(\ell+1)$

Radial equation

$$\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) - \ell(\ell+1) f = 0$$

Ansatz: $f(r) = r^\lambda$ for some constant λ

$$f' = \lambda r^{\lambda-1}$$

$$rf' = \lambda r^{\lambda+1}$$

$$\frac{d}{dr}(rf') = \lambda(\lambda+1) r^\lambda$$

$$\Rightarrow [\lambda(\lambda+1) - \ell(\ell+1)] r^\lambda = 0$$

must hold for all r, so $\lambda^2 + \lambda - \ell(\ell+1) = 0$ (quadratic in λ)

one solution is $\lambda = \ell$. Factor this as

$$(\lambda - \ell)(\lambda + \ell + 1) = 0$$

other solution is $\lambda = -\ell - 1$

$$\Rightarrow f(r) = A_\ell r^\ell + B_\ell r^{-\ell-1}$$

Equation in θ

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) + \ell(\ell+1)g - \frac{m^2}{\sin^2\theta}g = 0$$

$$\frac{d^2g}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dg}{d\theta} + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]g = 0$$

Rewrite this equation in terms of variable $x = \cos\theta$
[not the x-coordinate!]

Observe: $\frac{dg}{d\theta} = \frac{dg}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{dg}{dx}$

$$\frac{d^2g}{d\theta^2} = \frac{d}{d\theta}\left(-\sin\theta\frac{dg}{dx}\right) = -\cos\theta\frac{dg}{dx} - \sin\theta\frac{d}{dx}\left(\frac{dg}{dx}\right)\underbrace{\frac{dx}{d\theta}}_{-\sin\theta}$$

$$= -\cos\theta\frac{dg}{dx} + \sin^2\theta\frac{d^2g}{dx^2}$$

Eqn becomes

$$\left(\sin^2\theta\frac{d^2g}{dx^2} - \cos\theta\frac{dg}{dx}\right) + \frac{\cos\theta}{\sin\theta}\left(-\sin\theta\frac{dg}{dx}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]g = 0$$

N.B $\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$

$$(1-x^2)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + \left[\ell(\ell+1) - \frac{m^2}{(1-x^2)}\right]g = 0$$

called the associated Legendre equation

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The solutions to this eqn are complicated
+ generally ill-behaved at $x = \pm 1$, ie $\theta = 0, \pi$ (NP + SP)

There exist well-behaved solutions when
m and ℓ are integers with $|m| \le \ell$.

The solutions are ~~ℓth order polynomials in x~~
called associated Legendre ~~polynomials~~ functions $P_\ell^m(x)$

Most general (well-behaved) solution of Laplace's eqn
w/ azimuthal symmetry

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) \underbrace{P_\ell^m(\cos\theta)\, e^{im\phi}}$$

proportional to
spherical harmonics

$Y_{\ell m}(\theta,\phi)$

for reference

$$Y_{\ell m} = \pm \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\; P_\ell^m(\cos\theta)\, e^{im\phi}$$

$$Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}}\; P_\ell(\cos\theta)$$

In this course, we will restrict our attention to solutions of <u>azimuthal symmetry</u>

ie independent of ϕ.

This means $m = 0$ so $h(\phi) = \text{constant}$

The equation in θ becomes

$$(1-x^2)\frac{d^2 g}{dx^2} - 2x\frac{dg}{dx} + \ell(\ell+1)\, g = 0$$

called Legendre's equation

When ℓ = nonnegative integer, the equation has solutions that are polynomial in x
(ℓ = order of the polynomial)
called Legendre polynomials $P_\ell(x)$.
We impose on these the boundary condition $P_\ell(1) = 1$.

Hence a separable, azimuthally-symmetric solution of $\nabla^2 \Phi = 0$ takes the form

$$\Phi(r,\theta) = \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$$

Because $P_\ell(\cos\theta)$ form a complete basis of functions of $0 \le \theta < \pi$
the most general solution is therefore

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$$

$$(1-x^2)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + \ell(\ell+1)\,g = 0$$

were it not for this term, the eqn would have the monomial solution $g \sim x^\ell$

Try $g = c_\ell x^\ell + c_{\ell-2}x^{\ell-2} + c_{\ell-4}x^{\ell-4} + \dots$

$$\frac{dg}{dx} = \ell c_\ell x^{\ell-1} + (\ell-2)c_{\ell-2}x^{\ell-3} + \dots$$

$$\frac{d^2g}{dx^2} = \ell(\ell-1)c_\ell x^{\ell-2} + (\ell-2)(\ell-3)c_{\ell-2}x^{\ell-4} + \dots$$

plug in and collect powers of x:

$$0 = x^\ell\,[\underbrace{\ell(\ell-1) - 2\ell + \ell(\ell+1)}_{0}]\,c_\ell$$

$$+ x^{\ell-2}\,[\ell(\ell-1)c_\ell + \{\underbrace{-(\ell-2)(\ell-3) - 2(\ell-2) + \ell(\ell+1)}_{4\ell-2}\}c_{\ell-2}]$$

$$+ x^{\ell-4}\,[(\ell-2)(\ell-3)c_{\ell-2} + \{\dots\}c_{\ell-4}] + \dots$$

$$\Rightarrow c_{\ell-2} = -\frac{\ell(\ell-1)}{4\ell-2}c_{\ell-2}, \qquad c_{\ell-4} = \dots c_{\ell-2}$$

recursion relations...

$$g = c_\ell\left[x^\ell - \frac{\ell(\ell-1)}{2(2\ell-1)}x^{\ell-2} + \dots\right]$$

Choose c_ℓ so that $P_\ell(1) = 1$

The first 5 Legendre polynomials are

ℓ	$P_\ell(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
	$\vdots$

observe $P_\ell(1)=1$

observe: either even or odd

Fun fact: Legendre polynomials are orthogonal

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx = 0 \qquad \text{if } \ell \neq \ell'$$

Normalization

$$\int_{-1}^{1} P_\ell^2(x)\,dx = \frac{2}{2\ell+1}$$

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0 & \text{if } \ell\neq\ell' \\ \frac{2}{2\ell+1} & \text{if } \ell=\ell' \end{cases}$$

(Next term in the expansion: not for class, but for the problem)

$$+ x^{l-4}\left[\underbrace{(l-2)(l-3)c_{l-2} - (l-4)(l-5)c_{l-4} - 2(l-4)c_{l-4}} + l(l+1)c_{l-4}\right]$$

$$(l-2)(l-3)c_{l-2} + (8l-12)c_{l-4}$$

$$\Rightarrow \quad c_{l-4} = \frac{-(l-2)(l-3)}{(8l-12)} c_{l-2}$$

$$P_l = c_l\left[x^l - \frac{l(l-1)}{(4l-2)} x^{l-2} + \frac{l(l-1)(l-2)(l-3)}{(4l-2)(8l-12)} x^{l-4} + \dots\right]$$

$$P_4 = c_4\left[x^4 - \frac{6}{7}x^2 + \frac{3}{35}\right]$$

$$P_5 = c_5\left[x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right]$$

$$= \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$