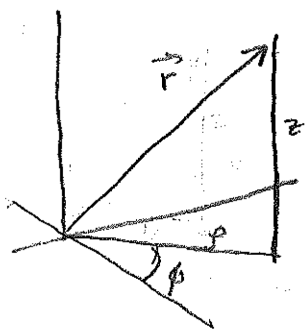
Laplace's eqn in cylindrical coordinates



$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$\rho > 0$
$0 \le \phi < 2\pi$

In this course, we will restrict our attention to z-independent solutions (e.g. boundary conditions uniform in z-direction)

so $\frac{\partial \Phi}{\partial z} = 0$ then $\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2}$

Try separable ansatz: $\Phi(\rho, \phi) = f(\rho)\, g(\phi)$

$$\frac{1}{\rho f}\frac{d}{d\rho}\left(\rho \frac{df}{d\rho}\right) + \frac{1}{\rho^2 g}\frac{d^2 g}{d\phi^2} = 0$$

Multiply by ρ^2 so each term is constant

$$\underbrace{\frac{\rho}{f}\frac{d}{d\rho}\left(\rho \frac{df}{d\rho}\right)}_{\text{constant.}} + \underbrace{\frac{1}{g}\frac{d^2 g}{d\phi^2}}_{\text{constant}} = 0$$

$g(\phi)$ should be a well-defined function of position, i.e. $g(\phi + 2\pi) = g(\phi)$

This requires $\frac{1}{g}\frac{d^2 g}{d\phi^2} = -m^2$ w/ m = integer

$$g(\phi) = C_m \cos(m\phi) + D_m \sin(m\phi), \quad m \neq 0$$

$$g(\phi) = C_0 + \underbrace{D_0}_{0}\,\phi, \quad m = 0$$

0 because $g(\phi + 2\pi) \neq g(\phi)$

3E-2

Radial eqn

$$\frac{\rho}{f}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) = m^2$$

First consider $m = 0$

$$\Rightarrow \frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) = 0$$

$$\Rightarrow \rho\frac{df}{d\rho} = A_0$$

$$df = \frac{A_0\, d\rho}{\rho}$$

$$f = A_0 \ln\rho + B_0$$

Next consider $m \neq 0$:

$$\rho\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) = m^2 f$$

$$\rho^2\frac{d^2f}{d\rho^2} + \rho\frac{df}{d\rho} = m^2 f \qquad \text{[ask for suggestions]}$$

Ansatz $f = \rho^\lambda$

$$\rho^2\lambda(\lambda-1)\rho^{\lambda-2} + \rho\lambda\rho^{\lambda-1} = m^2\rho^\lambda$$

$$\lambda^2 = m^2 \qquad \Rightarrow \lambda = \pm m$$

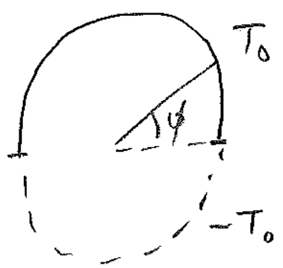
$$f = A_m\rho^m + B_m\rho^{-m}$$

Most general solution: combine separable solutions in all possible ways

$$\Phi(\rho,\phi) = \alpha_0 + \gamma_0\ln\rho + \sum_{m=1}^{\infty}\left[\alpha_m\rho^m\cos m\phi + \beta_m\rho^m\sin m\phi + \gamma_m\rho^{-m}\cos m\phi + \delta_m\rho^{-m}\sin m\phi\right]$$

Find steady-state temp distribution of a circular plate (insulated faces so no heat flow in z direction) w/ top half held at T_0 and bottom half held at $-T_0$

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$



steady-state
$\Downarrow$

$\nabla^2 T = 0$ and indep of z.

Region includes origin ($\rho = 0$)

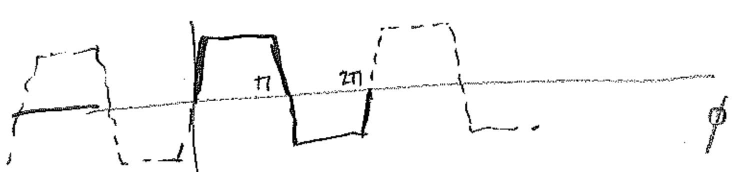
so $T(\rho, \phi) \xrightarrow[\rho\to 0]{}$ finite

$\Rightarrow \gamma_0 = 0, \quad \gamma_m = 0, \quad \delta_m = 0$

(N.B. in <u>ring</u> geometry, these can be nonzero)

$$T(\rho, \phi) = \alpha_0 + \sum_{m=1}^{\infty} \left[\alpha_m \rho^m \cos m\phi + \beta_m \rho^m \sin m\phi\right]$$

Apply b.c. $T(R, \phi) = \begin{cases} T_0 & 0 < \phi < \pi \\ -T_0 & \pi < \phi < 2\pi \end{cases}$



odd function, so only sines contribute: $\alpha_m = 0$

$T(R, \phi)$ averages to zero: $\alpha_0 = 0$ (no offset)

$$T(R, \phi) = \sum \beta_m R^m \sin(m\phi)$$

Fourier's trick: multiply by $\sin(n\phi)$ and integrate

$$\underbrace{\sum \beta_m R^m \underbrace{\int_0^{2\pi} \sin(m\phi)\sin(n\phi)\,d\phi}_{\frac{1}{2}\cdot(2\pi) \text{ if } m=n}}_{\pi\beta_n R^n} = \underbrace{\int_0^{\pi} T_0 \sin(n\phi)\,d\phi}_{-\frac{T_0}{n}\cos(n\phi)\Big|_0^{\pi}} + \underbrace{\int_{\pi}^{2\pi} (-T_0)\sin n\phi\,d\phi}_{\frac{T_0}{n}\cos(n\phi)\Big|_{\pi}^{2\pi}}$$

$$\pi\beta_n R^n = -\frac{T_0}{n}(\cos n\pi - 1) + \frac{T_0}{n}\Big(\underbrace{\cos(2\pi n)}_{1} - \cos \pi n\Big)$$

$$= \frac{2T_0}{n}(1-\cos(n\pi))$$

$$= \frac{4T_0}{n}\sin^2\left(\frac{\pi n}{2}\right)$$

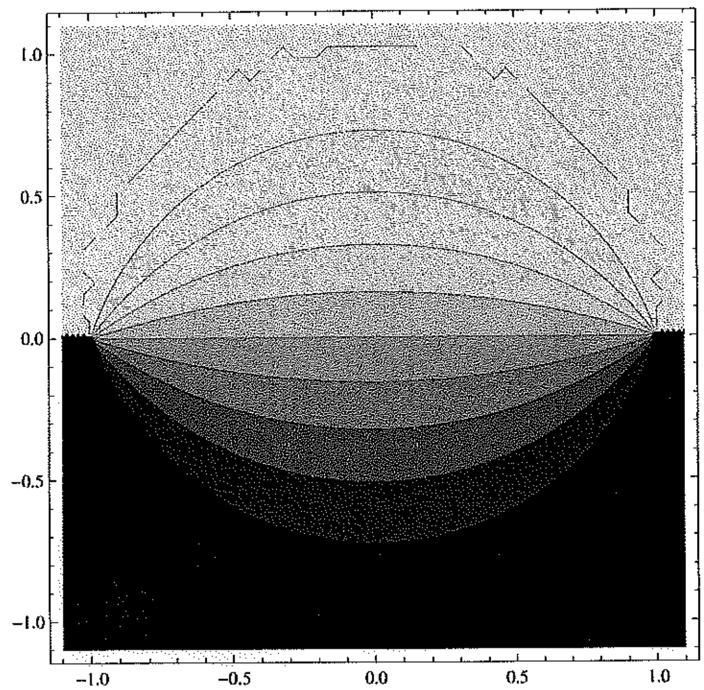
$$\beta_n = \frac{4T_0}{\pi n R^n}\sin^2\left(\frac{\pi n}{2}\right)$$

$$\Rightarrow T(\rho,\phi) = \sum_n \frac{4T_0}{\pi n}\sin^2\left(\frac{\pi n}{2}\right)\left(\frac{\rho}{R}\right)^n \sin(n\phi)$$

```
In[2]:= (* Steady-state temp distribution in
          circular plate upper half at 1 and lower half at 0 *)
        T[x_, y_, NN_] := If[x^2 + y^2 < 1,
          (4 / Pi) Sum[Im[(x + I y)^m] / m, {m, 1, NN, 2}], If[y > 0, 1, -1]]
```

```
In[10]:= ContourPlot[T[x, y, 2001], {x, -1.1, 1.1}, {y, -1.1, 1.1}]
```

Out[10]=




```
In[11]:= Plot3D[T[x, y, 2001], {x, -1.2, 1.2}, {y, -1.2, 1.2}]
```

Out[11]=