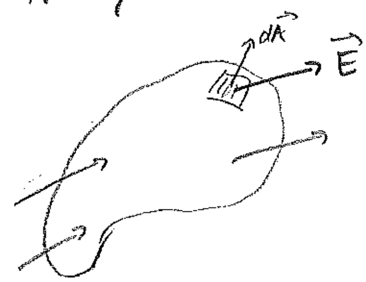
Flux

The electric flux through a closed surface Σ is defined as

$$\oint_\Sigma \vec{E}\cdot d\vec{A}$$



[flux = flow, but nothing is actually flowing]

We will now prove Gauss's law, which states

$$\oint_\Sigma \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

q_{enc} = charge enclosed by Σ.

Proof

Use Gauss's divergence theorem

$$\oint_\Sigma \vec{E}\cdot d\vec{A} = \int_V \vec{\nabla}\cdot\vec{E}\, dV$$

Consider field produced by a point charge at origin

$$\vec{E} = \frac{Kq}{r^2}\hat{r}$$

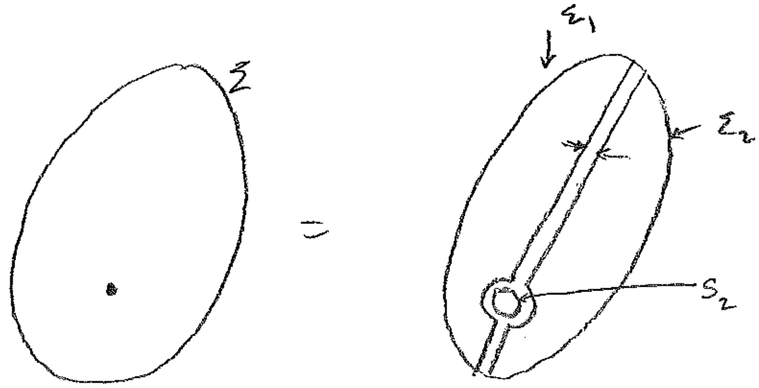
$$\vec{\nabla}\cdot\vec{E} = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}(\underbrace{r^2 E_r}_{Kq})}_{0} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \cancel{E_\theta}^{\,0}) + \frac{1}{r\sin\theta}\cancel{\frac{\partial E_\phi}{\partial\phi}}^{\,0}$$

$\vec{\nabla}\cdot\vec{E} = 0$, due to inverse square law.

$\hookrightarrow$ except at origin ($r=0$), because $\frac{0}{0}$ undefined

$\Rightarrow \oint_\Sigma \vec{E}\cdot d\vec{A} = 0$, provided Σ does not enclose the origin (where the charge is)

What if Σ includes the origin? "Avocado trick"



$$\oint_{\Sigma} \vec{E}\cdot dA = \oint_{\Sigma_1} \vec{E}\cdot d\vec{A} + \oint_{\Sigma_2} \vec{E}\cdot d\vec{A} + \oint_{S_2} \vec{E}\cdot d\vec{A}$$

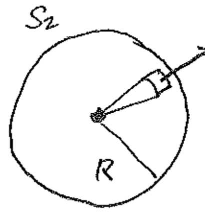
Σ ← any surface surrounding origin

S_2 ← sphere of radius R

- Interior surfaces cancel
- Σ_1 + Σ_2 do not enclose origin $\Rightarrow \oint_{\Sigma_{1,2}} \vec{E}\cdot d\vec{A} = 0$

$$\Rightarrow \oint_{\Sigma} \vec{E}\cdot d\vec{A} = \oint_{S_2} \vec{E}\cdot d\vec{A}$$

encloses origin



$$\vec{dA} = \hat{n}\, dA = \hat{r}\,(R\,d\theta)(R\sin\theta\, d\phi)$$

[look at reference sheets]

$$\oint_{S_2} \vec{E}\cdot\vec{dA} = \int \left(\frac{Kq}{R^2}\hat{r}\right)\cdot\left(\hat{r}\, R^2 \sin\theta\, d\theta\, d\phi\right)$$

$$= Kq \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{\underbrace{-\cos\theta\Big|_0^{\pi}}_{2}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$d\Omega \equiv \sin\theta\, d\theta\, d\phi$

$\int d\Omega = 4\pi$

area of unit sphere

$$= 4\pi K q = \frac{q}{\epsilon_0} \qquad \text{where } K \equiv \frac{1}{4\pi\epsilon_0}$$

independent of radius

For a point charge at origin:

$$\therefore \quad \oint_{\Sigma} \vec{E}\cdot\vec{dA} = \begin{cases} \frac{q}{\epsilon_0} & \text{if } \Sigma \text{ encloses point charge} \\ 0 & \text{if not} \end{cases}$$

Consider two charges:

$\cdot\, q_2$

superposition principle: $\vec{E} = \vec{E}_1 + \vec{E}_2$

$\cdot\, q_1$

$$\oint_\Sigma \vec{E}\cdot d\vec{A} = \underbrace{\oint_{\Sigma_1} \vec{E}_1 \cdot d\vec{A}}_{\frac{q_1}{\epsilon_0} \text{ if } \Sigma \text{ encloses } q_1} + \underbrace{\int_{\Sigma_2} \vec{E}_2 \cdot dA}_{\frac{q_2}{\epsilon} \text{ if } \Sigma \text{ encloses } q_2}$$

generalizes to arbitrary # of point charges

$$\boxed{\oint_\Sigma \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}} \leftarrow \text{total charge enclosed by } \Sigma$$

Gauss's law (integral form)

Assume also valid for continuous charge distribution

$$q_{enc} = \int_V \rho \, dV \qquad \text{where } V = \text{volume enclosed by } \Sigma$$
$$\text{ie } \partial V = \Sigma$$

$$\underbrace{\oint_{\partial V} \vec{E}\cdot d\vec{A}}_{\int_V \vec{\nabla}\cdot\vec{E}\, dV} = \int_V \frac{\rho}{\epsilon_0}\, dV$$

Since valid for arbitrary volumes, we must have

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}} \quad \text{Gauss's law (differential form)}$$

Thus, Coulomb + superposition $\Rightarrow$ $\vec{\nabla}\times\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

More generally

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

(Maxwell's eqns)

gravity

Point mass: $\vec{g} = -\frac{Gm}{r^2}\hat{r}$

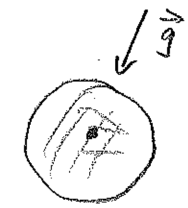
gravitational flux $\boxed{\oint_{\Sigma} \vec{g}\cdot d\vec{A} = -4\pi G M_{encl}}$

mass enclosed by Σ

(Gauss's law for gravity)

$\boxed{\vec{\nabla}\cdot\vec{g} = -4\pi G \rho}$ mass density

Consider a sphere of radius R and uniform mass density $\rho = \dfrac{M}{\left(\frac{4\pi}{3}R^3\right)}$

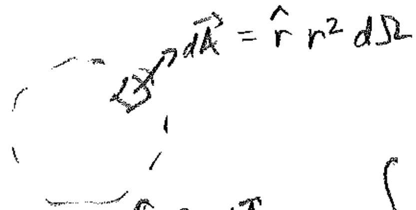


Symmetry $\Rightarrow$ $\vec{g}$ is radial

$\Rightarrow$ $|\vec{g}|$ only depends on r, not θ or ϕ

Thus $\vec{g} = g(r)\,\hat{r}$. Need to find $g(r)$.

Choose spherical surface Σ of radius r (either inside or outside sphere)



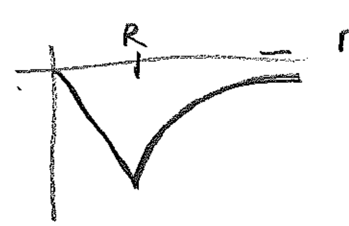
Flux $\oint \vec{g}\cdot d\vec{A} = \int g(r)\, r^2\, d\Omega = 4\pi r^2 g(r) = -4\pi G M_{encl}$

$$\Rightarrow g(r) = -\frac{G M_{encl}}{r^2}$$

If $r > R$, then $M_{encl} = M$

If $r < R$, then $M_{encl} = \int \rho\, dV = \rho\, \dfrac{4\pi r^3}{3} = M\left(\dfrac{r^3}{R^3}\right)$ because ρ is uniform

$$g(r) = \begin{cases} -\dfrac{GM}{r^2}, & r > R \\ -\dfrac{GM}{R^3}\, r, & r < R \end{cases}$$



(NB. harmonic oscillator)

Electrostatic field is conservative

$$\vec{\nabla}\times\vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}V$$

Gauss's law: $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$$\Rightarrow \quad \vec{\nabla}\cdot(-\vec{\nabla}V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$ Poisson equation

In a region of empty space, $\rho = 0$ so

$$\boxed{\nabla^2 V = 0}$$ Laplace's eqn

Laplace's eqns in different physical contexts

vibrating membrane $\eta(x,y,t)$:

$$0 = \frac{\partial^2 \eta}{\partial t^2} - v^2 \underbrace{\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right)}_{\nabla^2 \eta \text{ (two-dimensional)}} \xrightarrow[\text{static}]{} \nabla^2 \eta = 0$$

3d heat diffusion eqn

$$\frac{\partial T}{\partial t} = D \nabla^2 T \xrightarrow[\text{steady-state}]{} \nabla^2 T = 0$$

Poisson eqn of electrostatics

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \xrightarrow[\text{region of empty space}]{} \nabla^2 V = 0$$

Poisson eqn for gravity

$$\nabla^2 \Phi = +4\pi G \rho \xrightarrow[\text{empty}]{} \nabla^2 \Phi = 0$$

mass density