Moments of charge (or mass) distributions

(monopole, dipole, quadrupole, etc.)

Given a distribution of charges q_i at positions $\vec{r}_i$:

- Monopole moment = total charge $Q = \Sigma q_i$ or $\int dq'$ (scalar)
- dipole moment $\vec{p} = \Sigma q_i \vec{r}_i$ or $\int dq' \, \vec{r}\,'$ (vector)

(N.B. always express vectors in <u>Cartesian</u> components before summing or integrating, because unit vectors $\hat{x}, \hat{y}, \hat{z}$ don't vary in direction at different positions, unlike $\hat{r}, \hat{\theta}, \hat{\phi}$)

$$\vec{p} = \Sigma q_i (x_i \hat{x} + y_i \hat{y} + z_i \hat{z})$$

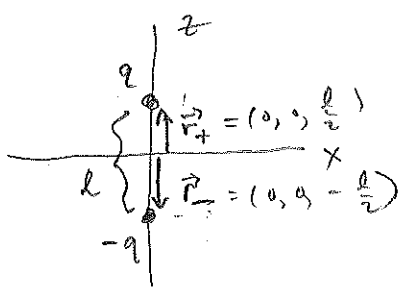
$$= \underbrace{(\Sigma q_i x_i)}_{p_x} \hat{x} + \underbrace{(\Sigma q_i y_i)}_{p_y} \hat{y} + \underbrace{(\Sigma q_i z_i)}_{p_z} \hat{z}$$

- quadrupole moment

$$\mathcal{Q} = \begin{pmatrix} \Sigma q_i x_i^2 & \Sigma q_i x_i y_i & \Sigma q_i x_i z_i \\ \Sigma q_i y_i x_i & \Sigma q_i y_i^2 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

(2nd rank tensor)

Example: electric dipole configuration



$$Q = q + (-q) = 0$$

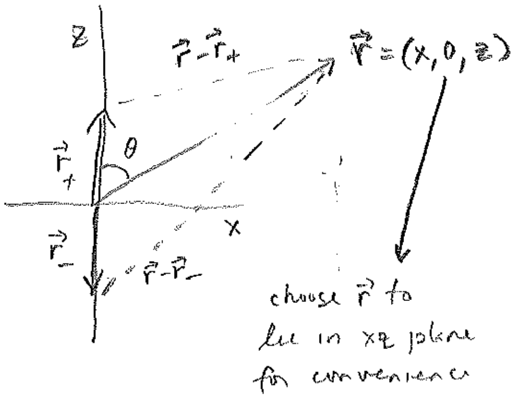
$$\vec{p} = q\vec{r}_+ + (-q)\vec{r}_-$$

$$= q(0, 0, \tfrac{l}{2}) - q(0, 0, -\tfrac{l}{2})$$

$$= (0, 0, ql)$$

$$= ql\,\hat{z}$$

Electric dipole field



$$\vec{E}(\vec{r}) = K \sum \frac{q_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

$$\vec{r}-\vec{r}_\pm = \left(x,\ 0,\ z \mp \frac{l}{2}\right)$$

$$|\vec{r}-\vec{r}_\pm|^2 = x^2 + \left(z \mp \frac{l}{2}\right)^2$$

$$= \underbrace{x^2+z^2}_{r^2} \mp zl + \frac{l^2}{4}$$

$$= r^2\left(1 \mp \frac{zl}{r^2} + \frac{l^2}{4r^2}\right)$$

$$\frac{1}{|\vec{r}-\vec{r}_\pm|^{3/2}} = \frac{1}{r^3}\left(1 \mp \frac{zl}{r^2} + \frac{l^2}{4r^2}\right)^{-3/2}$$

so far, everything is exact

We'll be interested in the far-field limit: $r \gg l$

Expand in small parameter $\frac{l}{r}$. Count powers of l.

$$(1+\epsilon)^\alpha = 1 + \alpha\epsilon + \frac{1}{2}\alpha(\alpha-1)\epsilon^2 + \ldots$$

$$\left(1 \mp \frac{zl}{r^2} + \frac{l^2}{4r^2}\right)^{-3/2} = 1 - \frac{3}{2}\left(\mp\frac{zl}{r^2} + \frac{l^2}{4r^2}\right) + \frac{15}{8}\left(\mp\frac{zl}{r^2} + \frac{l^2}{4r^2}\right)^2 + \ldots$$

$$= 1 \pm \frac{3zl}{2r^2} + \mathcal{O}(l^2)$$

this will be sufficient for dipole field

Dipole field

$$\vec{E} = Kq \left(\frac{(\vec{r}-\vec{r}_+)}{|\vec{r}-\vec{r}_+|^3} - \frac{(\vec{r}-\vec{r}_-)}{|\vec{r}-r_-|^3} \right)$$

$$E_x = Kq \left(\frac{x}{[x^2+(z-\frac{l}{2})^2]^{3/2}} - \frac{x}{[x^2+(z+\frac{l}{2})^2]^{3/2}} \right)$$

$$= \frac{Kqx}{r^3} \left(\left[1 - \frac{zl}{r^2} + \frac{l^2}{4r^2}\right]^{-3/2} - \left[1 + \frac{zl}{r^2} + \frac{l^2}{4r^2}\right]^{-3/2} \right)$$

Far field limit

$$\rightarrow \frac{Kqx}{r^3} \left(\left[1 + \frac{3zl}{2r^2} + \ldots\right] - \left[1 - \frac{3zl}{2r^2} + \ldots\right] \right)$$

$O(l^0)$ cancels so keep $O(l^1)$

$$= \frac{3Kq\,l\,zx}{r^5} = \frac{3Kp}{r^3}\left(\frac{z}{r}\right)\left(\frac{x}{r}\right) \quad \text{where } p = ql$$

$$E_y = 0$$

$$E_z = Kq \left(\frac{(z-\frac{l}{2})}{[x^2+(z-\frac{l}{2})^2]^{3/2}} - \frac{(z+\frac{l}{2})}{[x^2+(z+\frac{l}{2})^2]^{3/2}} \right)$$

$$= \frac{Kqz}{r^3} \left(\underbrace{\left[1 - \frac{zl}{r^2} + ..\right]^{-3/2} - \left[1 + \frac{zl}{r^2} + ..\right]^{-3/2}}_{\frac{3zl}{r^2}} \right)$$

$$- \frac{Kql}{2} \left(\underbrace{\left[1 - \frac{zl}{r^2} + ..\right]^{3/2} + \left[1 + \frac{zl}{r^2} + ..\right]^{-3/2}}_{2} \right)$$

already has one power of l, so only keep leading power

$$= \frac{Kql}{r^3}\left(\frac{3z^2}{r^2} - 1\right) = \frac{Kp}{r^3}\left(\frac{3z^2}{r^2} - 1\right)$$

In general, $\vec{E} \sim \frac{Kp}{r^3} \cdot$ (angular factors)

Recall $\begin{cases} z = r\cos\theta \\ x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \end{cases}$ $\quad y = 0 \Rightarrow \phi = 0$

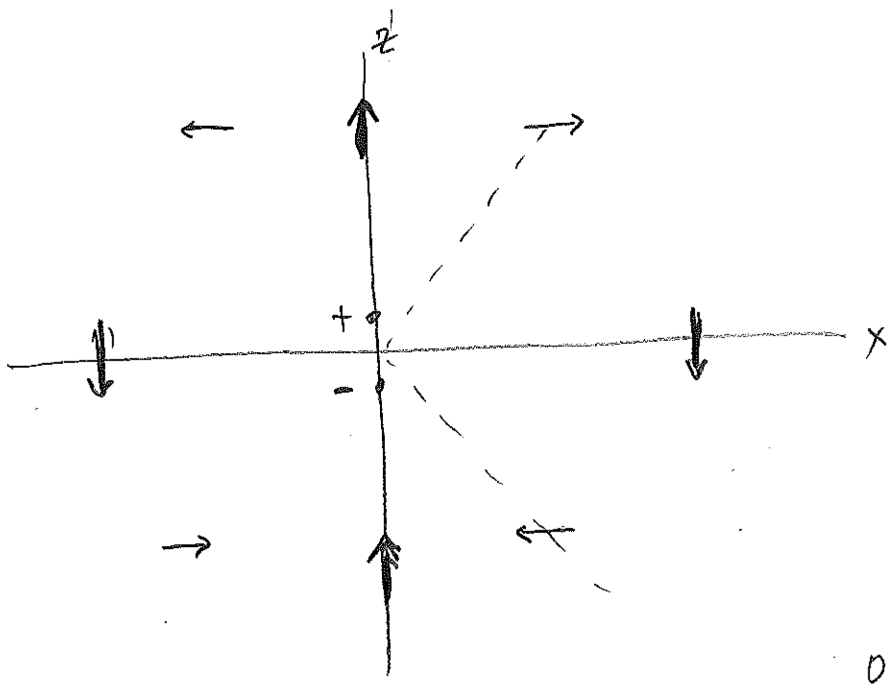
$$E_x = \frac{3Kp}{r^3}\left(\frac{z}{r}\right)\left(\frac{x}{r}\right) = \frac{3Kp}{r^3}\cos\theta\sin\theta\cos\phi$$

$$E_y = \frac{3Kp}{r^3}\cos\theta\sin\theta\sin\phi \quad \longleftarrow \text{generalizing to nonzero } \phi$$

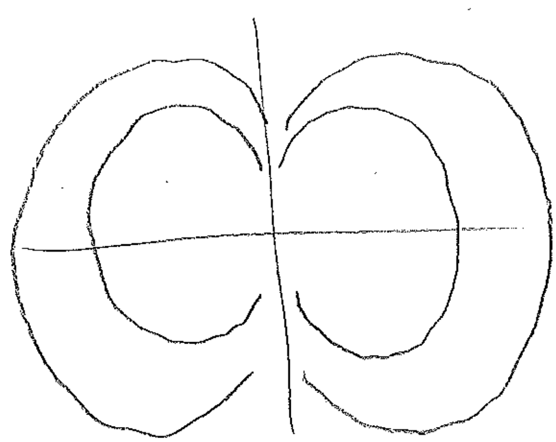
$$E_z = \frac{Kp}{r^3}\left(3\cos^2\theta - 1\right)$$

Note: $E_z = 0$ for $\cos^2\theta = \frac{1}{3}$

$\theta \approx 55^\circ$



(connect them w/ field lines)



Observe:

monopole field $\sim \dfrac{KQ}{r^2}$

dipole field $\sim \dfrac{Kp}{r^3}$

Even for more general charge configurations, far field limit depends only on Q, p, etc.

Redo calculation using electrostatic potential

$$V(\vec{r}) = \sum \frac{Kq_i}{|\vec{r}-\vec{r}_i|}$$

$$= \frac{Kq}{|\vec{r}-\vec{r}_+|} - \frac{Kq}{|\vec{r}-\vec{r}_-|}$$

$$= \frac{Kq}{r}\left[\frac{1}{\sqrt{1-\frac{zl}{r^2}+\frac{l^2}{4r^2}}} - \frac{1}{\sqrt{1+\frac{zl}{r^2}+\frac{l^2}{4r^2}}}\right]$$

Still exact. $-\vec{\nabla}V$ gives exact $\vec{E}$.

<u>Far-field limit of potential</u>

$$V \approx \frac{Kq}{r}\left[\left(1-\frac{zl}{2r^2}+O(l^2)\right)-\left(1+\frac{zl}{2r^2}+O(l^2)\right)\right]$$

$$= \frac{Kqlz}{r^3} = \frac{Kp}{r^2}\left(\frac{z}{r}\right) = \frac{Kp}{r^2}\cos\theta$$

(If $\vec{p}$ isn't aligned along z-axis, then becomes $V = \frac{K\vec{p}\cdot\vec{r}}{r^3}$)

Now compute $\vec{E} = -\vec{\nabla}V$

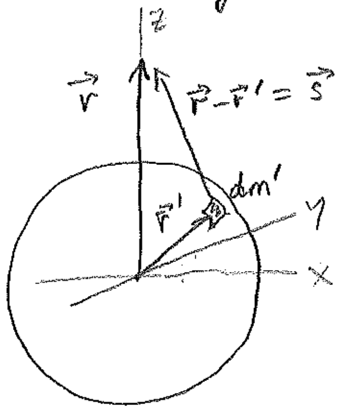
$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2} = \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

$$E_z = -\frac{\partial V}{\partial z} = -\left(\frac{Kp}{r^3} - \frac{3Kpz}{r^4}\underbrace{\frac{\partial r}{\partial z}}_{\frac{z}{r}}\right)$$

$$= \frac{Kp}{r^3}\left(\frac{3z^2}{r^2}-1\right) = \frac{Kp}{r^3}(3\cos^2\theta - 1) \checkmark$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{3Kpz}{r^4}\underbrace{\frac{\partial r}{\partial x}}_{\frac{x}{r}} = \frac{3Kp\,xz}{r^5} = \frac{3Kp}{r^3}\sin\theta\cos\theta \checkmark$$

Field caused by a spherical shell of mass m and radius R



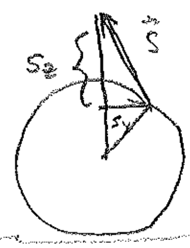
$$\vec{g}(\vec{r}) = -G\int \frac{dm'\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$= -G\int \frac{dm'\,\vec{s}}{s^3}$$

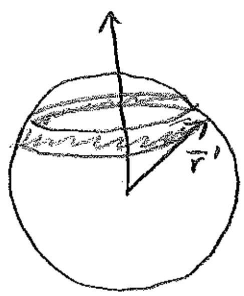
Decompose $\vec{s}$ into Cartesian components!

spherical symmetry of shell $\Rightarrow$ cancellation of x + y components of $\vec{g}$

$$g_z(\vec{r}) = -G\int \frac{dm'\,s_z}{s^3}$$



All points on a ring of mass contribute equally to g_z

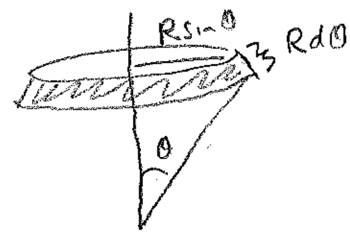


$$dm' = \sigma\, dA'$$

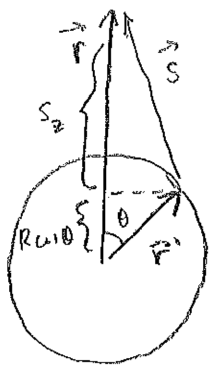
$$\sigma = \frac{\text{mass}}{\text{area}} = \frac{m}{4\pi R^2}$$

$$dA' = (2\pi R\sin\theta)\,R\,d\theta$$

$$\Rightarrow dm' = \frac{m}{2}\sin\theta\, d\theta$$



$$g_z(\vec{r}) = -\frac{Gm}{2}\int_{\theta=0}^{\pi} \frac{s_z\,\sin\theta\, d\theta}{s^3}$$



$$s^2 = (\vec{r}-\vec{r}')^2 = r^2 + \vec{r}'^2 - 2\vec{r}\cdot\vec{r}'$$
$$= r^2 + R^2 - 2rR\cos\theta$$

$$s_z = r - R\cos\theta$$

$$g_z = -\frac{Gm}{2}\int_0^{\pi} \frac{(r - R\cos\theta)\sin\theta\, d\theta}{(r^2+R^2-2rR\cos\theta)^{3/2}}$$

Try using s rather than θ as variable.

$$\cos\theta = \frac{r^2+R^2-s^2}{2rR}$$

$$-\sin\theta\, d\theta = -\frac{s\,ds}{rR}$$

$$s_z = r - \left(\frac{r^2+R^2-s^2}{2r}\right) = \frac{r^2-R^2+s^2}{2r}$$

$$g_z = -\frac{Gm}{2}\int_{s=r-R}^{r+R} \left(\frac{r^2-R^2+s^2}{2r}\right)\left(\frac{s\,ds}{rR}\right)\frac{1}{s^3}$$

$$= -\frac{Gm}{4r^2R}\int_{r-R}^{r+R}\left[\frac{r^2-R^2}{s^2} + 1\right]ds$$

$$= -\frac{Gm}{4r^2R}\left[\frac{(R^2-r^2)}{s} + s\right]\Bigg|_{r-R}^{r+R}$$

$$= -\frac{Gm}{4r^2R}\Big[\underbrace{(R-r) - (-R-r) + (r+R) - (r-R)}_{4R}\Big] = -\frac{Gm}{r^2}$$

Shell of mass behaves as a point mass at the center of the sphere

What if point is inside the shell.

only change is that limits of integration go from $R-r$ to $r+R$

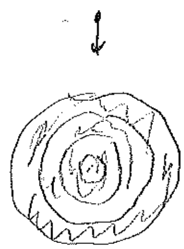
$$g_z = -\frac{Gm}{4r^2R}\left[\frac{R^2-r^2}{s} + s\right]\Bigg|_{R-r}^{r+R}$$

$$= -\frac{Gm}{4r^2R}\left[(R-r)-(R+r) + (r+R)-(R-r)\right]$$

$$= 0$$

Contributions of shell all cancel out

Solid sphere of mass: use superposition



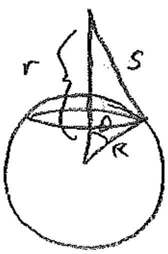
$$g_z = -\frac{Gm_{enc}}{r^2}$$

If inside the sphere, only mass interior to the point contributes

Problem

Gravitational potential due to uniform sphere

Use $\Phi = -G\int \frac{dm'}{|\vec{r}-\vec{r}'|}$



ⓐ shell of radius R and mass m.

As in class, spherical symmetry.

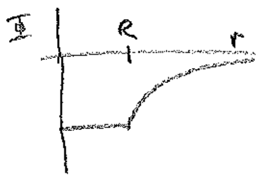
$\Rightarrow$ ring of mass $dm' = \frac{m}{2}\sin\theta\, d\theta$

$$|\vec{r}-\vec{r}'| = s$$

$$\cos\theta = \frac{r^2-R^2-s^2}{2rR} \Rightarrow \sin\theta\, d\theta = \frac{s\, ds}{rR}$$

$$\Phi(r) = -G\int \frac{\frac{m}{2}\sin\theta\, d\theta}{s} = -\frac{Gm}{2rR}\int_{r-R}^{r+R} ds = \boxed{-\frac{Gm}{r} \text{ for } r>R}$$

$$= -\frac{Gm}{2rR}\int_{R-r}^{R+r} ds = \boxed{-\frac{Gm}{R} \text{ for } r<R}$$



ⓑ For $r>R$, all shells are interior, so $\boxed{\Phi = -\frac{GM}{r},\ r>R}$

For $r<R$, integrate over shells inside & outside r: $\rho = \frac{M}{(\frac{4}{3}\pi R^3)}$

The mass of a shell of radius r' & thickness dr' is $4\pi\rho r'^2 dr' = \frac{3M}{R^3}r'^2 dr'$

$$\Phi(r) = \int_0^r \frac{3M}{R^3}r'^2 dr'\left(-\frac{G}{r}\right) + \int_r^R \frac{3M}{R^3}r'^2 dr'\left(-\frac{G}{r'}\right)$$

$$= -\frac{GM}{R^3}r^2 \quad -\frac{3GM}{2R^3}(R^2-r^2) = \boxed{-\frac{3GM}{2R} + \frac{GM}{2R^3}r^2,\quad r<R}$$

ⓒ $g = -\frac{\partial\Phi}{\partial r} = \begin{cases} -\frac{GM}{r^2}, & r>R \\ -\frac{GM}{R^3}r, & r<R \end{cases}$