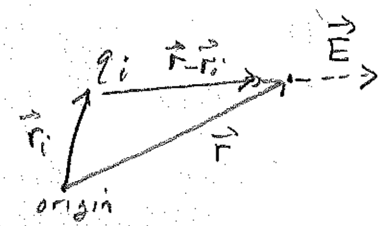
Electrostatic fields

A stationary point charge q_i at $\vec{r}_i$ produces an electric field at $\vec{r}$

$$\vec{E}(\vec{r}) = \frac{Kq_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} \qquad K = \frac{1}{4\pi\epsilon_0}$$



(Coulomb law) $\left[\begin{array}{l}\text{radial} \\ \text{inverse square}\end{array}\right]$

Superposition principle: a collection of point charges produces the vector sum

$$\vec{E}(\vec{r}) = K \sum_i \frac{q_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|}$$

A continuous distribution of charge produce

$$\vec{E}(\vec{r}) = K \int \frac{dq'(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \qquad \text{(integrate over position of charges)}$$

line of charge: $dq' = \lambda\, dl'$ $\qquad \lambda$ = charge/length
surface of charge $dq' = \sigma\, dA'$ $\qquad \sigma$ = charge/area
volume of charge $dq' = \rho\, dV'$ $\qquad \rho$ = charge/volume

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The force on a point charge q at $\vec{r}$ is proportional to the electric field produced by all other charges

$$\vec{F} = q\,\vec{E}(\vec{r})$$

Static gravitational fields

A stationary point mass m_i at $\vec{r}_i$ produces a gravitational field at $\vec{r}$

$$\vec{g}(\vec{r}) = -\frac{G m_i (\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} \qquad m_i \bullet \qquad \longleftarrow \vec{g}(\vec{r})$$

(Newton's law of univ. gravitation) [attractive]

$$\vec{g}(\vec{r}) = -G \sum \frac{m_i (\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

$$\vec{g}(\vec{r}) = -G \int \frac{dm' \, (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

[Newton: can replace sphere of mass by a point mass at center]

Force on a point mass m at $\vec{r}$ is

$$\vec{F} = m\,\vec{g}(\vec{r})$$

[2 other forces: strong & weak nuclear.
Not inverse square ⇒ short range ⇒ QM]

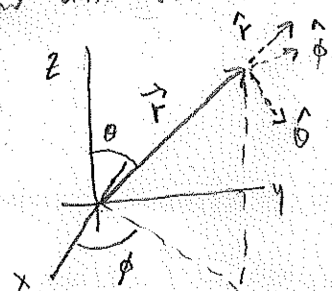
We will now show that the <u>electrostatic</u> force is conservative by proving $\vec{\nabla} \times \vec{E} = 0$.

(Faraday's law: $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ but no changing magnetic fields.)

First consider a static point charge at origin

$$\vec{E}(\vec{r}) = \frac{Kq\,\vec{r}}{|\vec{r}|^3} = \frac{Kq\,\hat{r}}{r^2} \quad \text{where } \hat{r} = \frac{\vec{r}}{r}$$

Review unit vectors in spherical coordinates [Hand out reference sheets]



$\hat{r}, \hat{\theta}, \hat{\phi}$ point in direction of increasing r, θ, ϕ respectively

A vector field $\vec{A}(\vec{r})$ can be written $A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$

Curl in spherical coordinates:

$$\vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, E_\phi) - \frac{\partial E_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}(rE_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]\hat{\phi}$$

$\vec{E}$ is purely radial: $E_r = \frac{Kq}{r^2}$, $E_\theta = E_\phi = 0$

$\Rightarrow \vec{\nabla} \times \vec{E} = 0$

- Static charge not at origin: $\vec{\nabla} \times \vec{E} = 0$ [shift coordinates]

- arbitrary distribution of charges: $\vec{\nabla} \times \vec{E} = 0$.

(superposition principle plus linearity of curl)

$\Rightarrow$ Electrostatic force obeys $\vec{\nabla} \times \vec{E} = 0$

$\vec{E}(\vec{r})$ is conservative, so it can be expressed as the gradient of a scalar field, called the electrostatic potential $V(\vec{r})$

$$\vec{E} = -\vec{\nabla} V \quad \Rightarrow \vec{E} \text{ field perpendicular to equipotential surfaces}$$

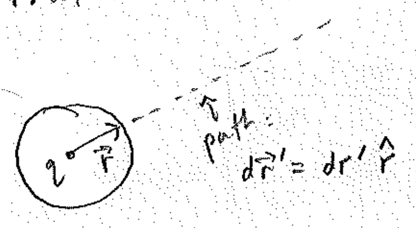
$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{r}' \qquad [\text{independent of path!}]$$

For a localized distribution of charges can choose $V(\infty) = 0$

Potential at $\vec{r}$ due to a point charge q at origin:

$$V(\vec{r}) = \frac{Kq}{|\vec{r}|}$$

Proof:
$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \left(\frac{Kq}{r'^2}\hat{r}\right) \cdot \left(dr'\,\hat{r}\right)$$



$$= -\int_{\infty}^{r} \frac{Kq}{r'^2}\,dr'$$

$$= \frac{Kq}{r'}\Big|_{\infty}^{r} = \frac{Kq}{r}$$

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point charge q_i at $\vec{r}_i$ produces potential

$$V(\vec{r}) = \frac{Kq_i}{|\vec{r}-\vec{r}_i|}$$

superposition principle also holds for potential

$$V(\vec{r}) = \sum_i \frac{Kq_i}{|\vec{r}-\vec{r}_i|} \quad \text{or} \quad K\int \frac{dq'}{|\vec{r}-\vec{r}'|}$$

$$\vec{E} = -\vec{\nabla} V \Rightarrow -\vec{\nabla}\left(\sum \frac{Kq_i}{|\vec{r}-\vec{r}_i|}\right) = \sum\left(-\nabla \frac{Kq_i}{|\vec{r}-\vec{r}_i|}\right) = \sum \vec{E}_i$$

Similarly, a static gravitational field $\vec{g}$ is conservative

$$\vec{\nabla} \times \vec{g} = 0 \qquad \text{so} \qquad \vec{g} = -\vec{\nabla}\Phi$$

where Φ = gravitational potential

$$\Phi(\vec{r}) = -G \sum \frac{m_i}{|\vec{r}-\vec{r}_i|} \quad \text{or} \quad -G \int \frac{dm'}{|\vec{r}-\vec{r}\,'|}$$

Finally: $\vec{F} = q\vec{E}$

$$= q(-\vec{\nabla}V)$$

$$= -\vec{\nabla}(qV)$$

$$= -\vec{\nabla}U$$ where $U = qV$ = electrostatic potential energy

$$\vec{F} = m\vec{g}$$

$$= m(-\vec{\nabla}\Phi)$$

$$= -\vec{\nabla}(m\Phi)$$

$$= -\vec{\nabla}U$$ $U = m\Phi$ = gravitational potential energy