<u>Review of vector differential operators</u>

del: $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

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grad: $\vec{\nabla} U = \hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}$ (scalar → vector)

div: $\vec{\nabla}\cdot\vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$ (vector → scalar)

Laplacian $= \text{div grad}$: $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$ (scalar → scalar)

curl: $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)$ (vector → vector)

$$+ \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)$$

$$+ \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

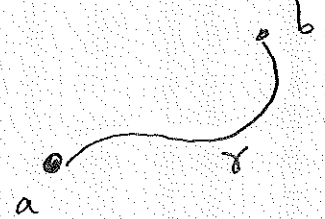
Later we'll learn two identities

div · curl = 0 ie $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$ for any $\vec{F}$

curl · grad = 0 ie $\vec{\nabla}\times(\vec{\nabla} U) = 0$ or U

Work done by forces

Suppose an object moves from point a to point b along some path γ.



Suppose some force $\vec{F}(x,y,z)$ acts on the object (e.g. grav., electrostatics, friction)

The work done by the force on the object is

$$W = \int_{a\,[\gamma]}^{b} \vec{F} \cdot d\vec{r} \quad \text{where} \quad d\vec{r} = \text{infinitesimal displacement along the path}$$

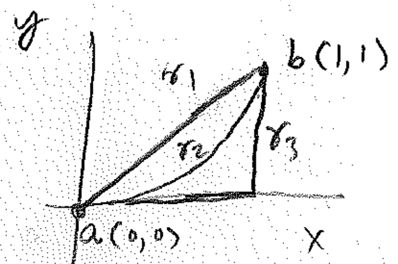
- The line integral depends on $\vec{F}$ and on the endpts a, b.
- It may or may not also depend on the path γ;
- If it does not, the force is called conservative

(We'll see this is equivalent to our earlier definition of a conservative force as $\vec{F} = -\vec{\nabla} U$)

In Cartesian coordinates $d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$

$$\vec{F} \cdot d\vec{r} = F_x\,dx + F_y\,dy + F_z\,dz$$

Example $\vec{F} = \begin{pmatrix} x^2-y^2 \\ 2xy \\ 0 \end{pmatrix} \Rightarrow W = \int_a^b (x^2-y^2)\,dx + 2xy\,dy$



[Note to self: don't put arrows on paths. Direction specified by endpoints]

Work along γ_1: $y = x \Rightarrow dy = dx$

$$W_1 = \int_0^1 (x^2-x^2)\,dx + 2x \cdot x\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

Work along γ_2: $y = x^2 \Rightarrow dy = 2x\,dx$

$$W_2 = \int_0^1 (x^2-x^4)\,dx + 2x(x^2)\,2x\,dx = \left(\frac{x^3}{3} + \frac{3x^5}{5}\right)\Big|_0^1 = \frac{14}{15}$$

Work along γ_3:

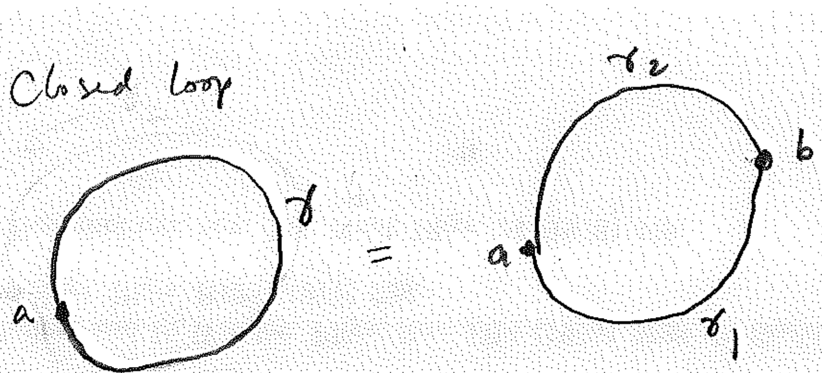
$$W_3 = \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r} + \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r}$$

$y = 0$, $dy = 0$ $\qquad$ $x = 1$, $dx = 0$

$$= \int_0^1 (x^2-0)\,dx + \int_0^1 2 \cdot 1 \cdot y\,dy = \frac{x^3}{3}\Big|_0^1 + y^2\Big|_0^1 = \frac{4}{3}$$

Since work depends on the path, this force is not conservative

Closed loop



A line integral around a closed loop is denoted

$$\oint_{[\gamma]} \vec{F} \cdot d\vec{r} = \int_{a\,[\gamma_1]}^{b} \vec{F} \cdot d\vec{r} + \int_{b\,[\gamma_2]}^{a} \vec{F} \cdot d\vec{r}$$

[direction is denoted by endpoints]

counterclockwise path

$$= \int_{a\,[\gamma_1]}^{b} \vec{F} \cdot d\vec{r} - \int_{a\,[\gamma_2]}^{b} \vec{F} \cdot d\vec{r}$$

[need specify direction for closed loops]

$$= W_1 - W_2$$

If $\vec{F}$ is conservative, then $W_1 = W_2$, so $\oint_{\gamma} \vec{F} \cdot d\vec{r}$ for any loop

Conversely if $\oint_{\gamma} \vec{F} \cdot d\vec{r}$ for any loop, then $\vec{F}$ is cons.

[argument]

Let's now prove Stokes's theorem

$$\oint_{\Sigma} (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = \oint_{\partial\Sigma} \vec{F}\cdot d\vec{r}$$

Σ ↑ arbitrary surface $\qquad$ $\partial\Sigma$ ↑ boundary of surface

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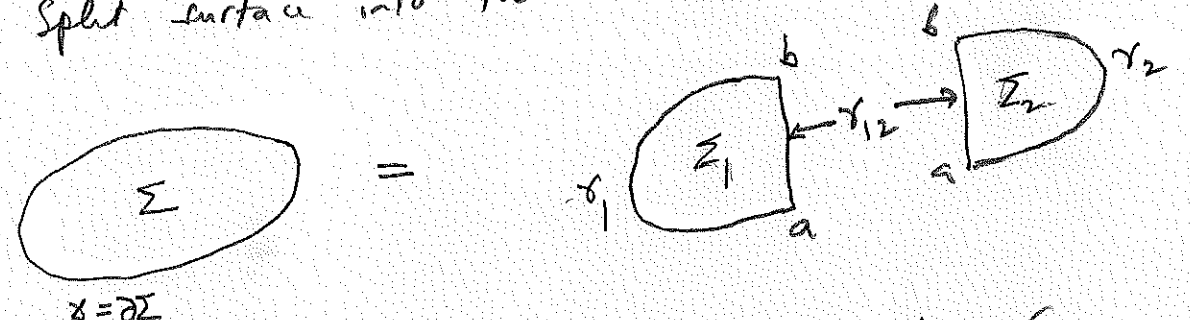
Split surface into two



Evaluate line integrals for each subsurface

$$\oint_{\partial\Sigma_1} \vec{F}\cdot d\vec{r} = \int_{b\,[\gamma_1]}^{a} \vec{F}\cdot d\vec{r} + \int_{a\,[\gamma_{12}]}^{b} \vec{F}\cdot d\vec{r}$$

equal + opposite

$$\oint_{\partial\Sigma_2} \vec{F}\cdot d\vec{r} = \int_{a\,[\gamma_2]}^{b} \vec{F}\cdot d\vec{r} + \int_{b\,[\gamma_{12}]}^{a} \vec{F}\cdot d\vec{r}$$

Therefore

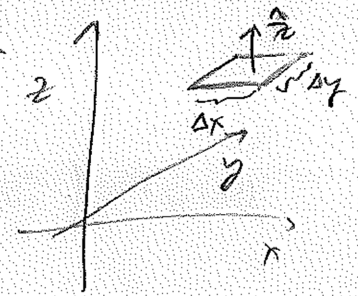
$$\oint_{\partial\Sigma_1} \vec{F}\cdot d\vec{r} + \oint_{\partial\Sigma_2} \vec{F}\cdot d\vec{r} = \oint_{[\gamma]} \vec{F}\cdot d\vec{r}$$

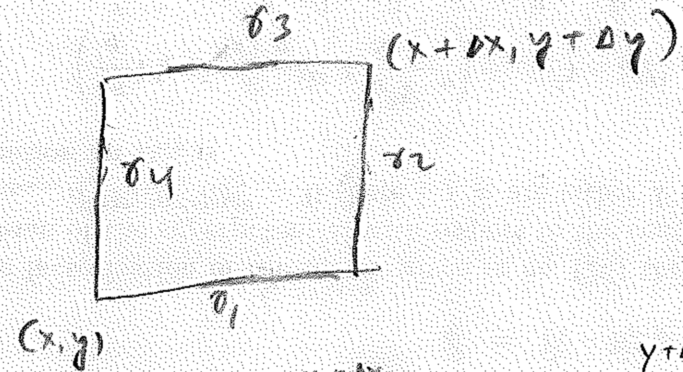
Keep subdividing 

$$\oint_{\gamma} \vec{F}\cdot d\vec{r} = \sum_{} \oint_{\text{tiny loop}} \vec{F}\cdot d\vec{r}$$

Consider tiny square loop in xy-plane

$\Rightarrow \Delta\vec{A} = \hat{z}\,\Delta x\,\Delta y$





$$\oint_{\square} \vec{F}\cdot d\vec{r} = \int_x^{x+\Delta x} F_x(\tilde{x},y)\,d\tilde{x} + \int_y^{y+\Delta y} F_y(x+\Delta x,\tilde{y})\,d\tilde{y}$$

counterclockwise

$$+ \int_{x+\Delta x}^{x} F_x(\tilde{x},y+\Delta y)\,d\tilde{x} + \int_{y+\Delta y}^{y} F_y(x,\tilde{y})\,d\tilde{y}$$

$$= \int_x^{x+\Delta x} \underbrace{\left[F_x(\tilde{x},y) - F_x(\tilde{x},y+\Delta y)\right]}_{-\frac{\partial F_x}{\partial y}(\tilde{x},y)\,\Delta y} d\tilde{x} + \int_y^{y+\Delta y} \underbrace{\left[F_y(x+\Delta x,\tilde{y}) - F_y(x,\tilde{y})\right]}_{\frac{\partial F_y}{\partial x}(x,\tilde{y})\,\Delta x} d\tilde{y}$$

$$\approx -\frac{\partial F_x}{\partial y}(x,y)\,\Delta y \underbrace{\int_x^{x+\Delta x} d\tilde{x}}_{\Delta x} + \frac{\partial F_y}{\partial x}(x,y)\,\Delta x \underbrace{\int_y^{y+\Delta y} d\tilde{y}}_{\Delta y}$$

$$= \left(\frac{\partial F_y}{\partial y} - \frac{\partial F_x}{\partial y}\right)\Delta x\,\Delta y$$

$$= (\vec{\nabla}\times\vec{F})_z\,\Delta x\,\Delta y$$

$$= (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,\Delta x\,\Delta y$$

$$= (\vec{\nabla}\times\vec{F})\cdot\Delta\vec{A}$$

(valid for any orientation of loop)

Fine print (asked by Arav)

$$\int \frac{\partial F_x}{\partial x}(x,\tilde{y})\,\Delta x\, d\tilde{y}$$

$$= \Delta x \int \left[\frac{\partial F_x}{\partial x}(x,y) + \frac{\partial^2 F_x}{\partial x \partial y}(x,y)(\tilde{y}-y) \right] d\tilde{y}$$

$$= \Delta x\, \Delta y\, \frac{\partial F_x}{\partial x}(x,y) + \Delta x \frac{\partial^2 F_x}{\partial x \partial y} \underbrace{\int_y^{y+\Delta y} (\tilde{y}-y)\, d\tilde{y}}$$

$$\frac{\tilde{y}^2}{2} - y\tilde{y} \Big|_y^{y+\Delta y}$$

$$= \frac{(y+\Delta y)^2}{2} - y(y+\Delta y) - \frac{y^2}{2} + y^2$$

$$= \cancel{\frac{y^2}{2}} + \cancel{y\Delta y} + \frac{1}{2}(\Delta y)^2 - \cancel{y^2} - \cancel{y\Delta y} - \cancel{\frac{y^2}{2}} + \cancel{y^2}$$

$$= \frac{1}{2}(\Delta y)^2$$

or change variable

$$\int_0^{\Delta y} \tilde{y}'\, d\tilde{y}'$$

$$= \frac{\tilde{y}'^2}{2} \Big|_0^{\Delta y}$$

$$= \frac{(\Delta y)^2}{2}$$

For a big loop

$$\oint_{[\gamma]} \vec F\cdot d\vec r = \sum_{\substack{\text{little}\\ \text{loops}}} \oint_{\square} \vec F\cdot d\vec r = \sum_{\substack{\text{little}\\ \text{squares}}} (\vec\nabla\times\vec F)\cdot d\vec A = \int_{\Sigma} (\vec\nabla\times\vec F)\cdot d\vec A$$

where Σ = surface whose boundary is γ

$$\boxed{\text{Stokes's theorem:} \quad \oint_{\gamma=\partial\Sigma} \vec F\cdot d\vec r = \int_{\Sigma} (\vec\nabla\times\vec F)\cdot d\vec A}$$

N.B. Direction of normal determined by r.h. rule
(fingers curl in direction of loop, thumb = $\hat n$)

Our earlier example $\vec F = \begin{pmatrix} x^2-y^2 \\ 2xy \\ 0 \end{pmatrix}$

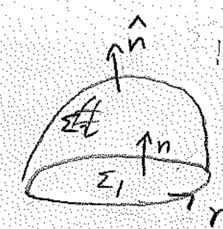
$$\vec\nabla\times\vec F = \begin{pmatrix} \frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4y \end{pmatrix}$$

$$\int_{\triangle} (\vec\nabla\times\vec F)\cdot d\vec A = \int_{x=0}^{1}\int_{y=0}^{x} (4y)\, dx\, dy = \int_0^1 dx \left(2y^2\Big|_0^x\right) = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$$

$$\oint_{\triangle} \vec F\cdot d\vec r = W_3 - W_1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3} \checkmark$$

If reverse direction $\Rightarrow -\frac{2}{3}$

$$\oint_{\gamma=\partial\Sigma} \vec{F}\cdot d\vec{r} = \int_{\Sigma} (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$$



side note

There are many surfaces whose boundary is γ

Does it matter which one we pick?

Consider the difference between these

$$\int_{\Sigma_2} (\vec{\nabla}\times\vec{F})\cdot d\vec{A} - \int_{\Sigma_1} (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$$

← maybe skip this since done in problem

flip $\hat{n}$ so it points outwards

$$= \oint_{\text{closed surface}} (\vec{\nabla}\times\vec{F})\cdot d\vec{A}$$

← Gauss's theorem

$$= \int_{\text{volume enclosed}} \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})\, dV = 0$$

Observe $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

$= 0$

vector calculus identity $\boxed{\text{div curl} = 0}$

No difference! Can choose any Σ whose boundary is γ

Suppose $\vec{F}$ can be expressed as $-\vec{\nabla} U$

Then $\vec{\nabla}\times\vec{F} = 0$ ($\vec{F}$ is curl-free or "irrotational")

Proof: $\vec{\nabla}\times(-\vec{\nabla}U) = \hat{x}\Big(\underbrace{\frac{\partial}{\partial y}\Big[-\frac{\partial U}{\partial z}\Big] - \frac{\partial}{\partial z}\Big[-\frac{\partial U}{\partial y}\Big]}_{\text{equality of mixed partial derivatives} \Rightarrow 0}\Big) + \hat{y}\cdots$

$= 0$

i.e. $\boxed{\text{curl grad} = 0}$

Equivalent statements about conservative fields
(a.k.a. "curl-free" or "irrotational" fields)

$\vec{F} = -\vec{\nabla} U$

$\Downarrow$ curl grad = 0

$\vec{\nabla} \times \vec{F} = 0$

$\Downarrow$ Stokes's

$\oint \vec{F} \cdot d\vec{r} = 0$

$\Downarrow$

$\int_\gamma \vec{F} \cdot d\vec{r}$ is independent of path

need to prove

Given conservative $\vec{F}$, define U by

(*) $U(\vec{r}) = U(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}$ ← needn't specify path because conservative

Since $U(\vec{r})$ only depends on endpoint $\vec{r}$, it is a well-defined function

To prove: (*) obeys $\vec{F} = -\vec{\nabla} U$

Consider $U(\vec{r} + \Delta\vec{r}) - U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}+\Delta\vec{r}} \vec{F} \cdot d\vec{r} - \int_{\vec{r}}^{\vec{r}_0} \vec{F} \cdot d\vec{r} = -\int_{\vec{r}}^{\vec{r}+\Delta\vec{r}} \vec{F} \cdot d\vec{r}$

$\approx -\vec{F}(\vec{r}) \cdot \Delta\vec{r}$ for small $\Delta\vec{r}$

But $U(x+\Delta x, y+\Delta y, z+\Delta z) = U(x,y,z) + \frac{\partial U}{\partial x} \cdot \Delta x + \frac{\partial U}{\partial y} \Delta y + \frac{\partial U}{\partial z} \Delta z + \ldots$

$U(\vec{r} + \Delta\vec{r}) = U(\vec{r}) + \vec{\nabla} U \cdot \Delta\vec{r}$

Comparing these gives $\vec{F} = -\vec{\nabla} U$ for arbitrary $\Delta\vec{r}$.

Equivalent statements about divergenceless fields

e.g. magnetic field

$\vec{B} = \vec{\nabla} \times \vec{A} \Leftrightarrow \vec{\nabla} \cdot \vec{B} = 0 \Leftrightarrow \oint \vec{B} \cdot d\vec{A} = 0 \Leftrightarrow \int_{\Sigma} \vec{B} \cdot d\vec{A}$ "

vector potential

div curl = 0

Gauss

for any closed surface

" is " independent of the surface (for a fixed boundary)

Summary

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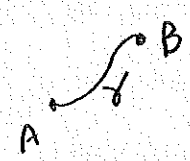
Gauss's thm $$\int_V (\vec{\nabla}\cdot\vec{J})\, dV = \oint_{\partial V} \vec{J}\cdot d\vec{A}$$ div curl = 0

Stokes's thm $$\int_A (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = \oint_{\partial A} \vec{F}\cdot d\vec{r}$$ curl grad = 0

$$\int_{A,\ \gamma}^{B} (\vec{\nabla} f)\cdot d\vec{r} = f(B) - f(A) = f\Big|_{\partial\gamma}$$



omit

generalized stokes

$$\int_{M} d\omega = \int_{\partial M} \omega$$

M ↑ d-dimensional

∂M ↑ (d-1) dim'l

$$d = \begin{cases} \vec{\nabla} & \text{grad} \\ \vec{\nabla}\times & \text{curl} \\ \vec{\nabla}\cdot & \text{div} \end{cases}$$

c

$$d \wedge d = 0 \Rightarrow \begin{array}{l} \text{curl grad} = 0 \\ \text{div curl} = 0 \end{array}$$