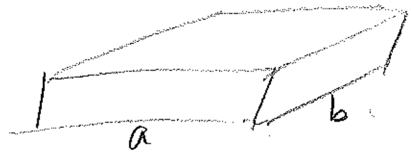
Consider rectangular plate w/ insulated top & bottom faces so no heat escapes from top/bottom, only sides

$\Rightarrow T$ is independent of z



$$\frac{\partial T}{\partial t} = D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

separable ansatz: $T(x,y,t) = f(x)\,g(y)\,h(t)$

$$f g \dot{h} = D(f'' g h + f g'' h)$$

$$\frac{\dot{h}}{h} = D\left(\frac{f''}{f} + \frac{g''}{g}\right)$$

$$\underbrace{\frac{f''}{f}}_{\text{only depends on } x} = \underbrace{\frac{1}{D}\frac{\dot{h}}{h} - \frac{g''}{g}}_{\text{indep of } x}$$

So $\frac{f''}{f} = \underbrace{-k_x^2}_{\text{const}}$ subscript! $\Rightarrow \frac{d^2 f}{dx^2} + k_x^2 f = 0 \Rightarrow f = \begin{cases} \cos k_x x \\ \sin k_x x \end{cases}$

$$\underbrace{\frac{g''}{g}}_{\text{only depends on } y} = \underbrace{\frac{1}{D}\frac{\dot{h}}{h} - \frac{f''}{f}}_{\text{indep of } y}$$

$$\frac{g''}{g} = -k_y^2 \Rightarrow \frac{d^2 g}{dy^2} + k_y^2 g = 0 \Rightarrow g = \begin{cases} \cos k_y y \\ \sin k_y y \end{cases}$$

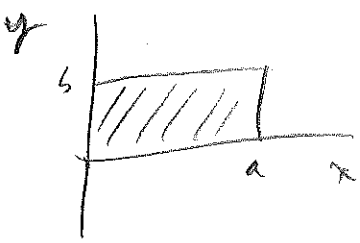
$$\frac{\dot{h}}{h} = D\left(\frac{f''}{f} + \frac{g''}{g}\right) = -D(k_x^2 + k_y^2)$$

$$h(t) = e^{-D(k_x^2+k_y^2)t}$$

so

$$T(x,y,t) = \begin{Bmatrix} \cos k_x x \\ \sin k_x x \end{Bmatrix} \begin{Bmatrix} \cos k_y y \\ \sin k_y y \end{Bmatrix} e^{-D(k_x^2+k_y^2)t}$$

<u>Boundary conditions</u>



Assume $T=0$ on all edges when $t \geq 0$.

$\Rightarrow$ cosine terms are absent

$\sin(k_x a) = 0 \quad \Rightarrow \quad k_x = \frac{n\pi}{a}$

$\sin(k_y b) = 0 \quad \Rightarrow \quad k_y = \frac{m\pi}{b}$

[N.B. n + m needn't be same integer!]

$$T(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-D\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)t}$$

Initial conditions

Let plate have initial temp. distribution $T_0(x,y)$

$$T(x,y,0) = \sum_{n,m} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) = T_0(x,y)$$

double Fourier series

multiply both sides by: $\sin\left(\frac{\tilde{n}\pi x}{a}\right) \sin\left(\frac{\tilde{m}\pi y}{b}\right)$ and integrate

$$\sum_{n,m} A_{nm} \underbrace{\int_0^a \sin\left(\frac{\tilde{n}\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx}_{\frac{a}{2}\delta_{\tilde{n}n}} \underbrace{\int_0^b \sin\left(\frac{\tilde{m}\pi y}{b}\right) \sin\left(\frac{m\pi y}{b}\right) dy}_{\frac{b}{2}\delta_{\tilde{m}m}}$$

$$= \frac{ab}{4} A_{\tilde{n}\tilde{m}}$$

so

$$A_{\tilde{n}\tilde{m}} = \frac{4}{ab} \int_0^a \int_0^b T_0(x,y) \sin\left(\frac{\tilde{n}\pi x}{a}\right) \sin\left(\frac{\tilde{m}\pi y}{b}\right) dx\, dy$$

Example: plate initially at uniform temp T_0
Then at $T=0$ at edges for $t \geq 0$.

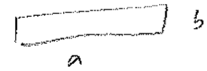
$$A_{nm} = \frac{4T_0}{ab} \underbrace{\int_0^a \sin\left(\frac{n\pi x}{a}\right) dx}_{\frac{a}{n\pi}\left(\underbrace{1-\cos n\pi}_{2\sin^2\frac{\pi n}{2}}\right)} \underbrace{\int_0^b \sin\left(\frac{m\pi y}{b}\right) dy}_{\frac{b}{m\pi}\left(\underbrace{1-\cos m\pi}_{2\sin^2\frac{\pi m}{2}}\right)}$$

$$= \frac{16 T_0}{\pi^2 nm} \sin^2\left(\frac{\pi n}{2}\right)\sin^2\left(\frac{\pi m}{2}\right) = \begin{cases} \frac{16T_0}{\pi^2 nm}, & n, m \text{ both odd} \\ 0, & \text{otherwise} \end{cases}$$

$\therefore$

$$T(x,y,t) = \frac{16T_0}{\pi^2} \sum_{\substack{n,m \\ \text{both odd}}} \frac{1}{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-D\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)t}$$

$$\simeq \frac{16T_0}{\pi^2} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) e^{-D\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)t} + \left(\begin{matrix}\text{faster} \\ \text{decaying} \\ \text{terms}\end{matrix}\right)$$



CNB if $a \gg b$ then $e^{-\frac{D\pi^2}{b^2}t}$
[= most heat escapes from top + bottom]

```
In[1]:= amp = (4 / Pi) ^2; mode[x_, y_, t_, nx_, ny_] :=
  (amp / nx / ny) * Sin[nx * Pi * x] * Sin[ny * Pi * y] * Exp[-t * (  (Pi nx) ^2 + (Pi ny) ^2) ];
partialsum[x_, y_, t_, m_] := Sum[Sum[mode[x, y, t, nx, ny], {nx, 1, m, 2}], {ny, 1, m, 2}];
Plot3D[partialsum[x, y, 0, 15], {x, 0, 1}, {y, 0, 1}, PlotRange → {{0, 1}, {0, 1}, {0, 1.4}}]
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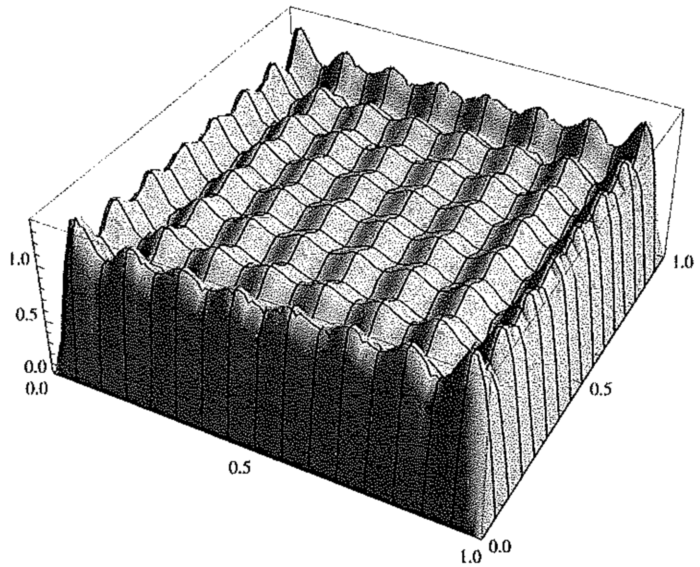


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Plot3D[partialsum[x, y, 0.001, 15], {x, 0, 1}, {y, 0, 1}]
```

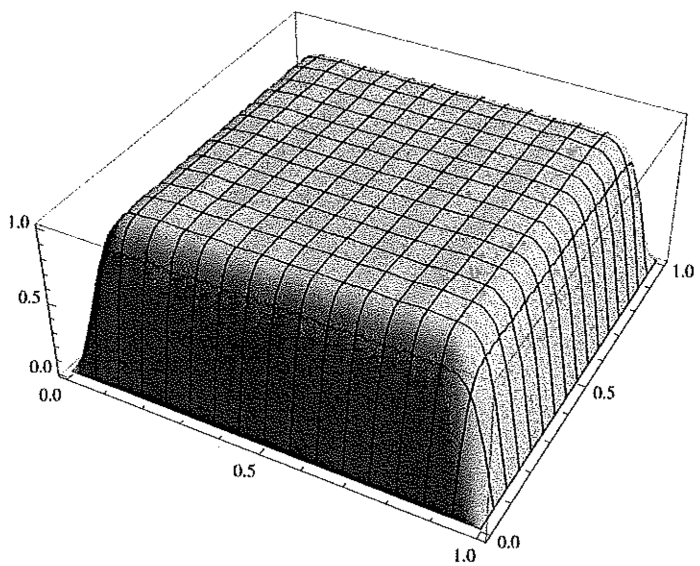

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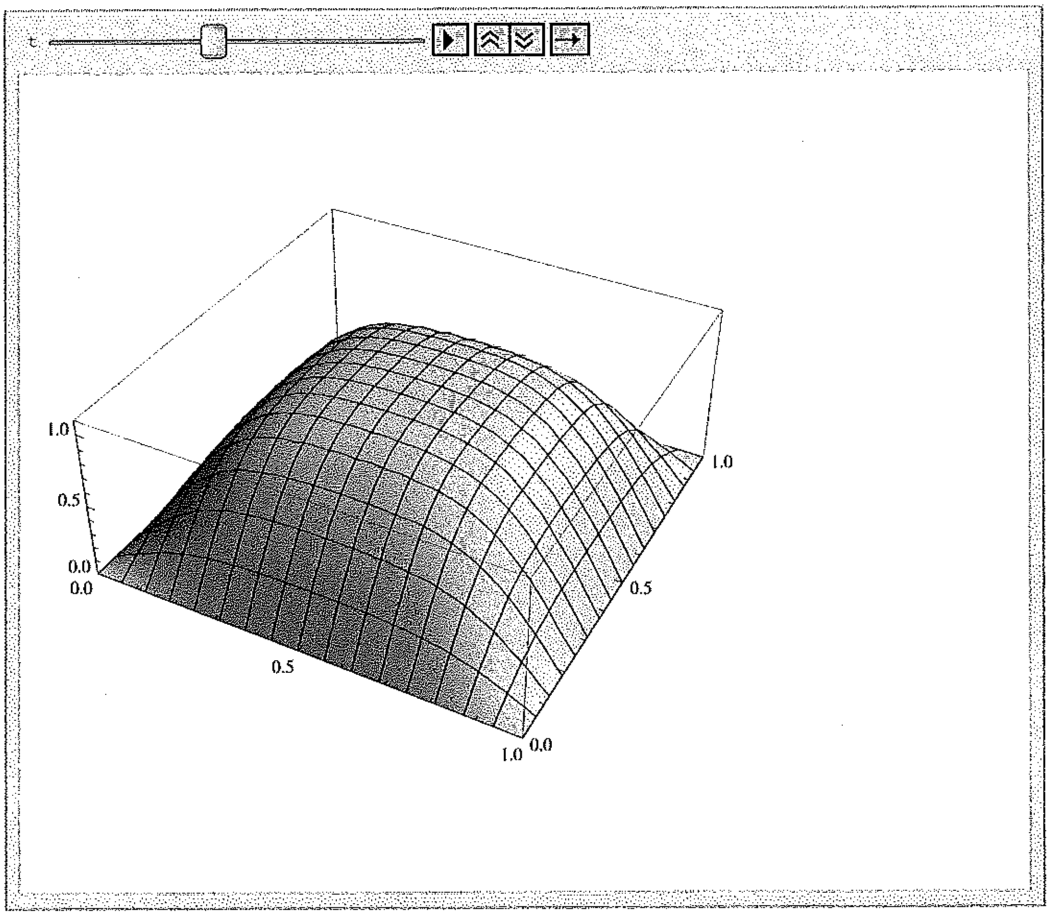
Animate[Plot3D[partialsum[x, y, t, 15], {x, 0, 1},
  {y, 0, 1}, PlotRange → {{0, 1}, {0, 1}, {0, 1.1}}], {t, 0, .03}]
```



(Used to be a problem) (done in class now)

(N12.6.99)

$$\frac{\partial T}{\partial t} = \frac{k}{c\rho}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$D = \frac{k}{c\rho} = \frac{0.93}{(0.0923)(8.960)} = 1.124535 \ \frac{cm^2}{s}$$

$$T(x,y,t) = f(x)\, g(y)\, h(t)$$

$$f(x) = \sin\left(\frac{\pi m x}{a}\right)$$

$$g(y) = \sin\left(\frac{\pi n y}{a}\right)$$

$$h(t) = e^{-\frac{D\pi^2}{a^2}(m^2+n^2)t}$$

$$T(x,y,t) = \sum a_{mn} \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{a}\right) e^{-\frac{D\pi^2}{a^2}(m^2+n^2)t}$$

$$T(x,y,0) = T_0 \quad \Rightarrow \quad a_{mn} = \frac{16 T_0}{\pi^2 mn} \quad \text{for } m+n \text{ both odd}$$

(a) $$T(x,y,t) = \sum_{m,n \text{ odd}} \frac{16 T_0}{\pi^2 mn} \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{a}\right) e^{-\frac{D\pi^2}{a^2}(m^2+n^2)t}$$

$-\frac{1}{2}$ if didn't indicate m, n are odd in the sum (even if indicated it earlier)

(b) First term $$T(x,y,t) \approx \frac{16 T_0}{\pi^2} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) e^{-\frac{2D\pi^2}{a^2}t}$$

At $T_0 = 100$ C

At $x = y = \frac{a}{2}$ $\Rightarrow$ $$\frac{1600}{\pi^2} e^{-\frac{2D\pi^2}{a^2}t} = \frac{1600}{\pi^2} e^{-(2.21974E-3 s^{-1})t}$$

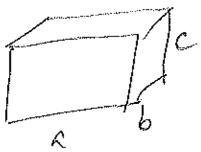
At $t = 10$ min, exponent is 1.3318 $\Rightarrow$ $42.798^\circ \Rightarrow 42.8^\circ$

At $t = 20$ min exponent is 2.6637 $\Rightarrow$ $11.298^\circ \Rightarrow 11.3^\circ$

Return to 3d

$$\frac{\partial T}{\partial t} = D\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)$$

Consider steady state distribution in a rectangular solid w/ faces held at different temps.



steady-state $\Rightarrow \frac{\partial T}{\partial t} = 0 \Rightarrow$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} = 0 \qquad \text{(laplace)}$$

Separable ansatz $T(x,y,z) = f(x)\,g(y)\,h(z)$

$$\Rightarrow \underbrace{\frac{f''}{f}}_{c_x} + \underbrace{\frac{g''}{g}}_{c_y} + \underbrace{\frac{h''}{h}}_{c_z} = 0$$

3 cases

① $c_x < 0$ $\qquad c_x = -k^2 \Rightarrow f'' + k^2 f = 0 \Rightarrow f = \begin{cases}\cos(kx)\\ \sin(kx)\end{cases}$

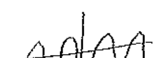
② $c_x = 0$ $\qquad f'' = 0 \Rightarrow f = \begin{cases} x \\ 1 \end{cases}$

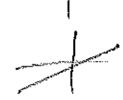
③ $c_x > 0$ $\qquad c_x = \alpha^2 \Rightarrow f'' - \alpha^2 f = 0 \Rightarrow f = \begin{cases} e^{\alpha x} \\ e^{-\alpha x}\end{cases}$ or $\begin{cases}\cosh \alpha x \\ \sinh \alpha x\end{cases}$

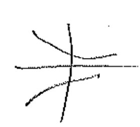
$f'' \gtrless 0$ concave up/down

$f \gtrless 0$ above/below axis

$c = \frac{f''}{f} \gtrless 0$ concave (toward / away from) axis

①  (concave toward axis)

②  (not concave)

③  (concave away from axis)

Since $c_x + c_y + c_z = 0$, solution must have different curvatures in different directions, e.g. $(+,+,-)$ or $(+,-,-)$ or $(+,-,0)$ or $(0,0,0)$

If boundary condition is $T=0$ at both ends in a certain direction, then curvature must be toward the axis $(c<0)$.
[because otherwise fcn can't pass thru zero twice]

- Suppose the solid is held at $T=0$ on all the side faces

Then
$$c_x = -k_x^2 \qquad k_x = \frac{\pi n}{a} \qquad f(x) = \sin\frac{\pi n x}{a}$$
$$c_y = -k_y^2 \qquad k_y = \frac{\pi m}{b} \qquad g(y) = \sin\frac{\pi m y}{b}$$

$$\Rightarrow c_z = \alpha^2 = k_x^2 + k_y^2 \Rightarrow h(z) = A\cosh \alpha z + B \sinh \alpha z$$

- Suppose bottom face also held at $T=0 \Rightarrow A = 0$

$$\Rightarrow T(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh(\alpha_{nm} z)$$

$$\alpha_{nm} = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

- Suppose top face is held at $T = T_0$

$$T(x,y,c) = \sum_{n,m} \underbrace{B_{nm} \sinh(\alpha_{nm} c)}_{A_{nm}} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) = T_0$$

[Same double Fourier series we just did]

Previously we found $A_{nm} = \begin{cases} \frac{16 T_0}{\pi^2 nm} & \text{for } n,m \text{ both odd} \\ 0 & \text{otherwise} \end{cases}$

$$\therefore \quad T(x,y,z) = \frac{16 T_0}{\pi^2} \sum_{\substack{n,m \\ \text{odd}}} \frac{1}{nm} \frac{\sinh(\alpha_{nm} z)}{\sinh(\alpha_{nm} c)} \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi m y}{b}\right)$$

[Show plots]

If $T_0 = 0$, then $T(x,y,z) = 0$ everywhere.
In general, if $T = 0$ on ∂, then Laplace's eqn $\Rightarrow T = 0$ in interior

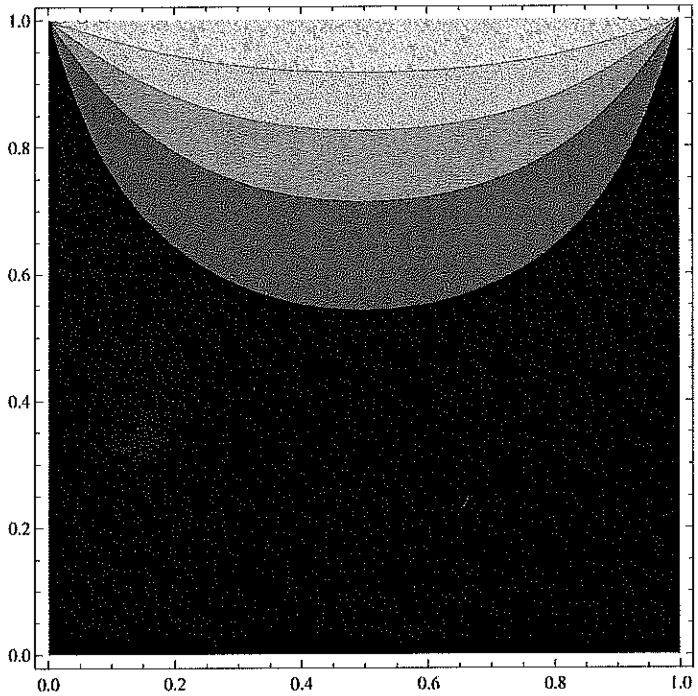
- If different faces held at diff temps.
  solve problem by adding solutions for only one face nonzero

2014

```
(* Steady-state temp distribution in cube with z=1 at T=1,
and other faces at T=0 *)
F[x_, y_, z_, m_] := (16 / Pi^2) * Sum[Sum[
      Sin[nx Pi x] Sin[ny Pi y]
       Sinh[Sqrt[nx^2 + ny^2] Pi z] / Sinh[Sqrt[nx^2 + ny^2] Pi] / nx / ny,
      {nx, 1, m, 2}], {ny, 1, m, 2}];
```

```
In[16]:= (* Cross-section in xz plane at y=1/2 *)
ContourPlot[F[x, .5, z, 55], {x, 0, 1}, {z, 0, 1}]
```

Out[16]=

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In[14]:= Plot3D[F[x, .5, z, 55], {x, 0, 1}, {z, 0, 1}]
```
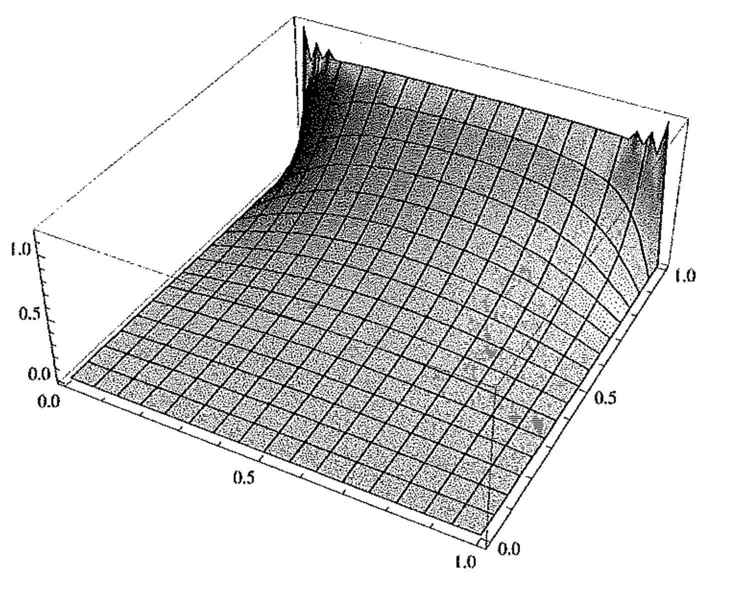

Out[14]=



In[10]:=
```
(* Cross-section in xy plane at z=1/2 *)
ContourPlot[F[x, y, 0.5, 15], {x, 0, 1}, {y, 0, 1}]
```

Out[10]=