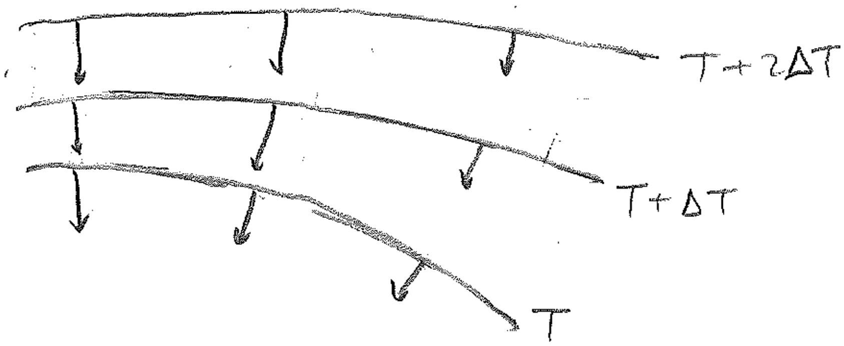
Thermal diffusion (3 dimensions)



Isotherms
= surfaces of const temperature

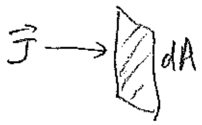
thermal energy flows in direction perpendicular to isotherms
(ie along gradient)

$$\text{grad } T = \vec{\nabla} T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial y} \right)$$

$$\vec{J}_h = -k \vec{\nabla} T \quad \text{(Fourier's law)}$$

$$J_h = |\vec{J}| = \frac{\text{thermal energy}}{(\text{area})(\text{time})}$$

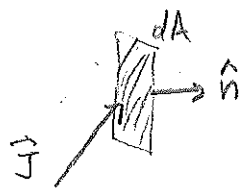
Let I be a current of anything
and $\vec{J}$ = current density (current / area)



$$I = J\,dA$$

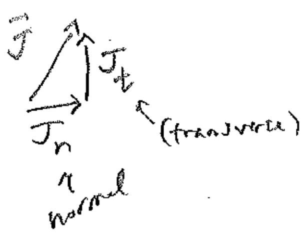
through an area element perpendicular to $\vec{J}$

What if area element is not perpendicular to $\vec{J}$?



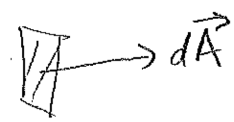
Define $\hat{n}$ as unit vector normal to area element

J_n = component normal to area $= \vec{J}\cdot\hat{n}$



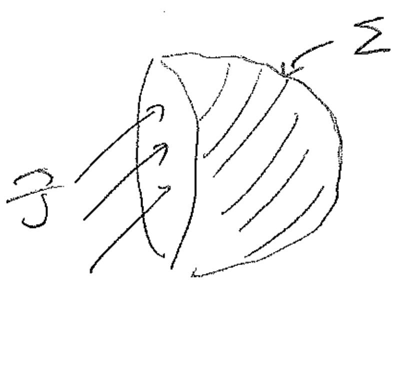
$$I = J_n\,dA = \vec{J}\cdot\hat{n}\,dA$$

Define <u>oriented</u> area element $d\vec{A} = dA\,\hat{n}$



$$I = \vec{J}\cdot d\vec{A}$$

How much current through a surface Σ?



$$I = \int_{\Sigma} \vec{J}\cdot d\vec{A}$$

[cf water through a fish net]

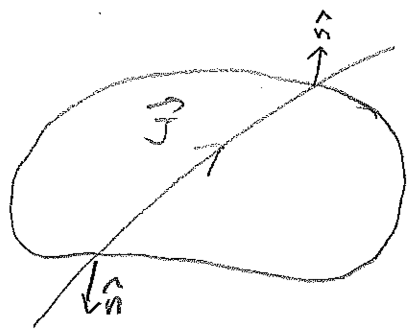
Consider a volume V w/ boundary ∂V

Net current flowing out of V is

$$\oint_{\partial V} \vec{J} \cdot d\vec{A} = \oint_{\partial V} \vec{J} \cdot \hat{n} \, dA$$

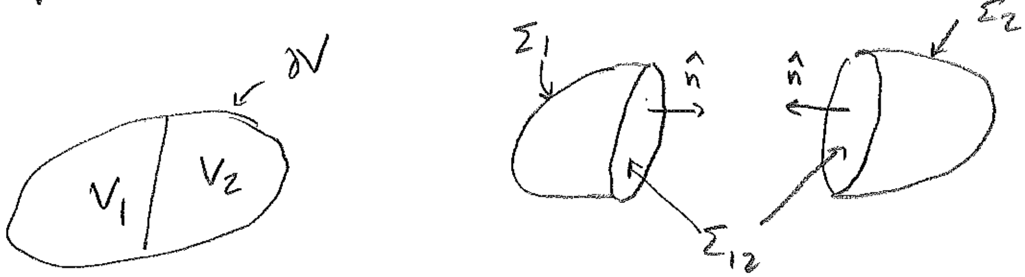
circle denotes closed surface (i.e. w/o boundary)

denotes outward normal



[some parts of surface contribute positively & some negatively]

Split volume into two pieces



energy flow out of each volume is

$$\oint_{\partial V_1} \vec{J}\cdot d\vec{A} = \int_{\Sigma_1} \vec{J}\cdot d\vec{A} + \int_{\Sigma_{12}} \vec{J}\cdot d\vec{A}$$

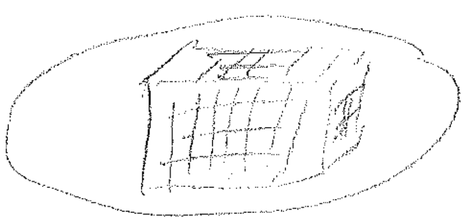
outward normal are opposite, so these are negative of each other

$$\oint_{\partial V_2} \vec{J}\cdot d\vec{A} = \int_{\Sigma_2} \vec{J}\cdot d\vec{A} + \int_{\Sigma_{12}} \vec{J}\cdot d\vec{A}$$

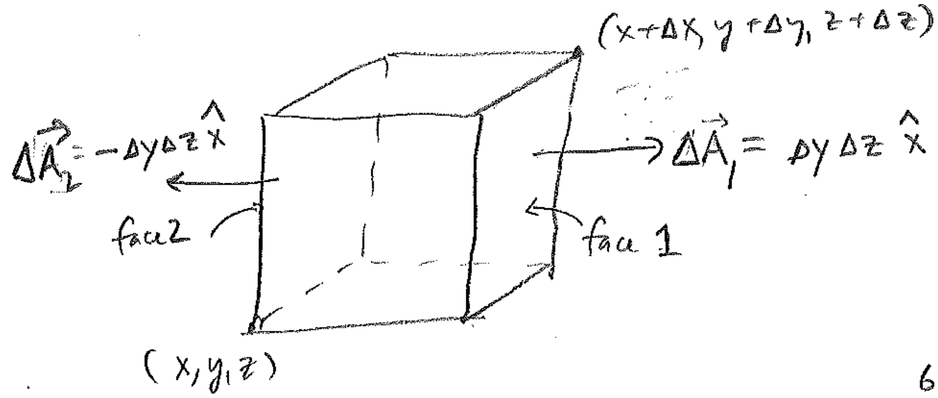
$$\oint_{\partial V_1} \vec{J}\cdot d\vec{A} + \oint_{\partial V_2} \vec{J}\cdot d\vec{A} = \int_{\Sigma_1} \vec{J}\cdot d\vec{A} + \int_{\Sigma_2} \vec{J}\cdot d\vec{A} = \int_{\partial V} \vec{J}\cdot d\vec{A}$$

Flow out of a volume = sum of flow out of each subvolume

$$\int_{\partial V} \vec{J}\cdot d\vec{A} = \sum_{\substack{\text{tiny}\\\text{cubes}}} \int_{\square} \vec{J}\cdot d\vec{A}$$



Tiny cube



Flow out of cube $\displaystyle\oint_{\partial V} \vec{J}\cdot d\vec{A} = \sum_{i=1}^{6} \vec{J}_i \cdot \Delta \vec{A}_i$

$$= \vec{J}_1 \cdot \hat{x}\, \Delta y \Delta z - \vec{J}_2 \cdot \hat{x}\, \Delta y \Delta z + (4 \text{ other faces})$$

[recall 3d Taylor expansion?]

$$= \underbrace{J_x(x+\Delta x, y, z)}_{J_x(x,y,z) + \frac{\partial J_x}{\partial x}\cdot \Delta x + \ldots} \Delta y \Delta z - J_x(x, y, z)\, \Delta y \Delta z + \ldots$$

$$= \underbrace{\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right)}_{\vec{\nabla}\cdot\vec{J} \quad \text{div } J \quad (\text{divergence})} \Delta x \Delta y \Delta z$$

$$\vec{\nabla}\cdot\vec{J} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \cdot \left(J_x \hat{x} + J_y \hat{y} + J_z \hat{z} \right)$$

For an arbitrary volume

$$\oint_{\partial V} \vec{J}\cdot d\vec{A} = \sum_{\text{cubes}} \oint_{\partial \square} \vec{J}\cdot d\vec{A} = \sum_{\text{cubes}} \vec{\nabla}\cdot\vec{J}\, \Delta x \Delta y \Delta z = \int_V \vec{\nabla}\cdot\vec{J}\, dV$$

Gauss's divergence theorem

$$\boxed{\oint_{\partial V} \vec{J}\cdot d\vec{A} = \int_V (\vec{\nabla}\cdot\vec{J})\, dV}$$

Recall ρ_h = thermal energy density

thermal energy in volume $= \int_V \rho_h \underbrace{dx\,dy\,dz}_{dV}$

Rate of change of thermal energy $= \int_V \left(\frac{\partial \rho_h}{\partial t}\right) dV$

If energy is flowing out of surface of V at rate $\oint_{\partial V} \vec{J}_h \cdot d\vec{A}$

then energy conservation implies

$$\int_V \left(\frac{\partial \rho_h}{\partial t}\right) dV = -\oint_{\partial V} \vec{J}_h \cdot d\vec{A}$$

[assuming no energy generation e.g. radioactivity or heating coils]

By Gauss's theorem

$$\int_V \left(\frac{\partial \rho_h}{\partial t}\right) dV = -\int_V \left(\vec{\nabla}\cdot\vec{J}_h\right) dV$$

Taking the volume to be infinitesimal

$$\frac{\partial \rho_h}{\partial t} + \vec{\nabla}\cdot\vec{J}_h = 0 \quad \text{at each point}$$

(continuity equation in 3d) (Recall $\frac{\partial \rho_h}{\partial t} + \frac{\partial J}{\partial x} = 0$ in 1 d)

Also applies to electrical charge density ρ + current density $\vec{J}$ by charge conservation

$$\frac{\partial \rho_h}{\partial t} + \vec{\nabla}\cdot\vec{J}_h = 0$$

Recall $d\rho_h = \underset{\uparrow \text{ specific heat}}{c}\,\rho_m\, dT$ ← mass density

and $\vec{J}_h = -k\vec{\nabla}T$ (Fourier's law)

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\Rightarrow \quad c\rho_m\frac{\partial T}{\partial t} - k\,\vec{\nabla}\cdot(\vec{\nabla}T) = 0$$

$$\vec{\nabla}\cdot(\vec{\nabla}T) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)$$

$$= \Big(\underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}_{=\vec{\nabla}\cdot\vec{\nabla} = \nabla^2 = \text{Laplacian operator}}\Big)T$$

Thus $\underbrace{\boxed{\frac{\partial T}{\partial t} = D\nabla^2 T}}_{\text{3d heat diffusion equation}}$ where thermal diffusivity $D = \frac{k}{c\rho_m}$

[Boyce DePrima call $D = \alpha^2$ but why squared?]

Steady-state $\Rightarrow \frac{\partial T}{\partial t} = 0 \Rightarrow \underbrace{\boxed{\nabla^2 T = 0}}_{\text{Laplace's eqn}}$