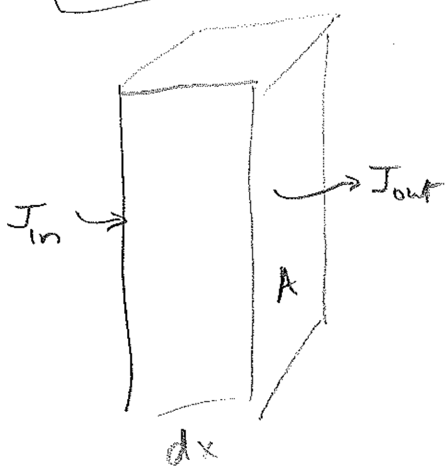
Thermal diffusion (one dimension)

Recall heat current density $J_h = \dfrac{(\text{energy})}{(\text{area})(\text{time})}$

Flow of thermal energy is driven by temperature difference

(1) $$\boxed{J_h = -k \frac{\partial T}{\partial x}}$$ (Fourier's law) k = thermal conductivity



If $J_{in} > J_{out}$, energy builds up in slab

Define thermal energy density $\rho_h = \dfrac{(\text{energy})}{(\text{volume})}$

Thermal energy in slab $= \rho_h A \, dx$

change in energy per time = net inflow of energy

$$\frac{\partial \rho_h}{\partial t} A \, dx = (J_{in} - J_{out}) A$$
$$= [J_h(x) - \underbrace{J_h(x+dx)}_{J_h(x) + \frac{\partial J_h}{\partial x} dx}] A = -\frac{\partial J_h}{\partial x} dx \, A$$

(2) $$\boxed{\frac{\partial \rho_h}{\partial t} = -\frac{\partial J_h}{\partial x}}$$ "continuity equation" (energy conservation)

Combine (1) + (2) $\Rightarrow$ (3) $$\boxed{\frac{\partial \rho_h}{\partial t} = k \frac{\partial^2 T}{\partial x^2}}$$

change in thermal energy is proportional to change in temperature

$$\Delta \rho_h \, A \, dx = C \, \Delta T$$

C = heat capacity of slab

define specific heat c = heat capacity per unit mass

Then $C = c \cdot \rho_m \cdot A \, dx$, where ρ_m = mass density

$$\underbrace{c}_{\left(\frac{\text{heat capacity}}{\text{mass}}\right)} \cdot \underbrace{\rho_m}_{\left(\frac{\text{mass}}{\text{vol}}\right)} \cdot \underbrace{A \, dx}_{(\text{volume of slab})}$$

$$\Rightarrow \quad \Delta \rho_h = c \rho_m \, \Delta T$$

$$(4) \quad \boxed{\frac{\partial \rho_h}{\partial t} = c \rho_m \frac{\partial T}{\partial t}}$$

Combine (3) + (4)

$$c \rho_m \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$$\boxed{\boxed{\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}}}$$ one dim'l heat diffusion eqn

where D = thermal diffusivity $= \frac{k}{c \rho_m}$

Not for class

Cu

$k = 0.93 \frac{\text{cal}}{\text{cm} \cdot \text{s} \cdot \text{C}}$

$c = 0.0923 \frac{\text{cal}}{\text{g} \cdot \text{C}}$

$\rho = 8.960 \text{ g/cm}^3$

$D = 1.12\text{E}{-4} \frac{\text{m}^2}{\text{s}}$

H_2O

$k = 0.58 \text{ J/m} \cdot \text{s} \cdot \text{C}$

$c = 4.185 \text{ J/g} \cdot \text{C}$

$\rho = 10^6 \text{ g/m}^3$

$D = 1.4\text{E}{-4} \frac{\text{m}^2}{\text{s}}$

Steady-state distribution in 1 dimension

means T depends on x, not t

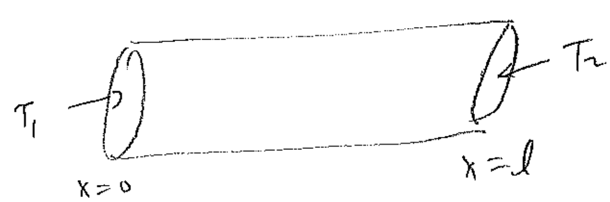
$$\frac{\partial T}{\partial t} = 0 \Rightarrow T = T(x)$$

Then diffusion equation $\Rightarrow \dfrac{d^2T}{dx^2} = 0$

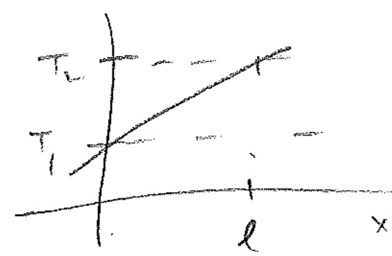
$$\frac{dT}{dx} = A$$

$$T = Ax + B$$

Use boundary conditions to determine A & B



$$T(x) = T_1 + (T_2 - T_1)x$$



Non steady state heat flow

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$

[2nd order linear PDE (2 indep vbls x, t)]

Try separable ansatz $T(x,t) = f(x)\, g(t)$

Strategy: ① substitute ansatz into PDE
② divide by ansatz + simplify
③ observe that each term is separately constant

$\dot{g} = \frac{dg}{dt}$

$f'' = \frac{d^2 f}{dx^2}$

① $f \dot{g} = D g f''$

② $\frac{\dot{g}}{g} = \frac{D f''}{f}$

③ $\underbrace{\frac{f''}{f}}_{\text{only depends on } x} = \underbrace{\frac{1}{D}\frac{\dot{g}}{g}}_{\text{indep of } x}$

$\hookrightarrow \therefore \frac{f''}{f} = \text{const}$

Is the constant positive or negative? B.C. will force it to be negative.

Let $\frac{f''}{f} = -k^2$

$\frac{\dot{g}}{g} = -k^2 D$

$\frac{dg}{dt} + k^2 D g = 0$

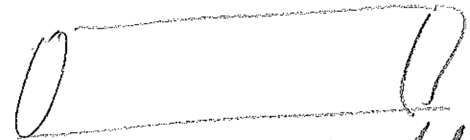
$\frac{d^2 f}{dx^2} + k^2 f = 0$

$g = 1 \cdot e^{-Dk^2 t}$

$f = A \cos kx + B \sin kx$

$\Rightarrow T(x,t) = (A \cos kx + B \sin kx)\, e^{-Dk^2 t}$ solves PDE

Apply boundary condition



Suppose both ends held at $T = 0$ for $t \geq 0$

$$T(0,t) = A\, e^{-D^2 k t} = 0 \quad \Rightarrow \quad A = 0$$

$$T(l,t) = B \sin(kl)\, e^{-Dk^2 t} = 0 \quad \Rightarrow \quad B = 0 \text{ or } kl = n\pi$$

$$k_n = \frac{n\pi}{l}$$

General solution

$$T(x,t) = \sum_{n=1}^{\infty} B_n \sin(k_n x)\, e^{-Dk_n^2 t}$$

Apply initial condition:

Suppose rod has initial temperature distribution $T_0(x)$

$$T(x,0) = \sum B_n \sin(k_n x) = T_0(x)$$

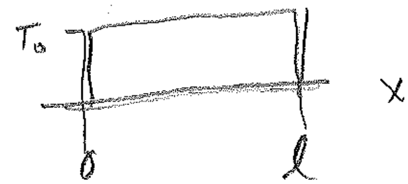
$\uparrow$ Fourier coefficients

$$B_n = \frac{2}{l} \int_0^l T_0(x) \sin(k_n x)\, dx$$

Suppose rod is heated to a uniform initial temp T_0

then at $t \geq 0$, both ends are held at $T=0$.

Assume sides are thermally insulated so no heat flows laterally



$$B_n = \frac{2}{\ell} T_0 \int_0^{\ell} \sin\left(\frac{\pi n x}{\ell}\right) dx$$

$$= \frac{2}{\ell} T_0 \left[-\frac{\ell}{n\pi} \cos\left(\frac{\pi n x}{\ell}\right)\right]\Big|_0^{\ell}$$

$$= \frac{2T_0}{n\pi}\left[1 - \cos(n\pi)\right]$$

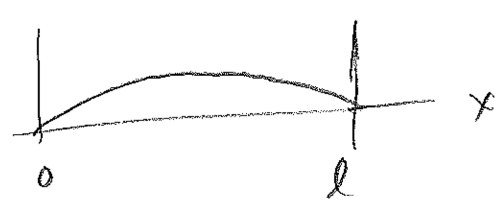
$$= \frac{4T_0}{\pi n} \sin^2\left(\frac{n\pi}{2}\right) = \begin{cases} \frac{4T_0}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$T_0(x) = \frac{4T_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\left(\frac{n\pi x}{\ell}\right)$$ → see composite square on Blackboard

$$= \frac{4T_0}{\pi}\left[\sin\left(\frac{\pi x}{\ell}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{\ell}\right) + \dots\right]$$

$$T(x,t) = \frac{4T_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\left(\frac{n\pi x}{\ell}\right) e^{-n^2\left(\frac{\pi^2 D}{\ell^2}\right)t}$$

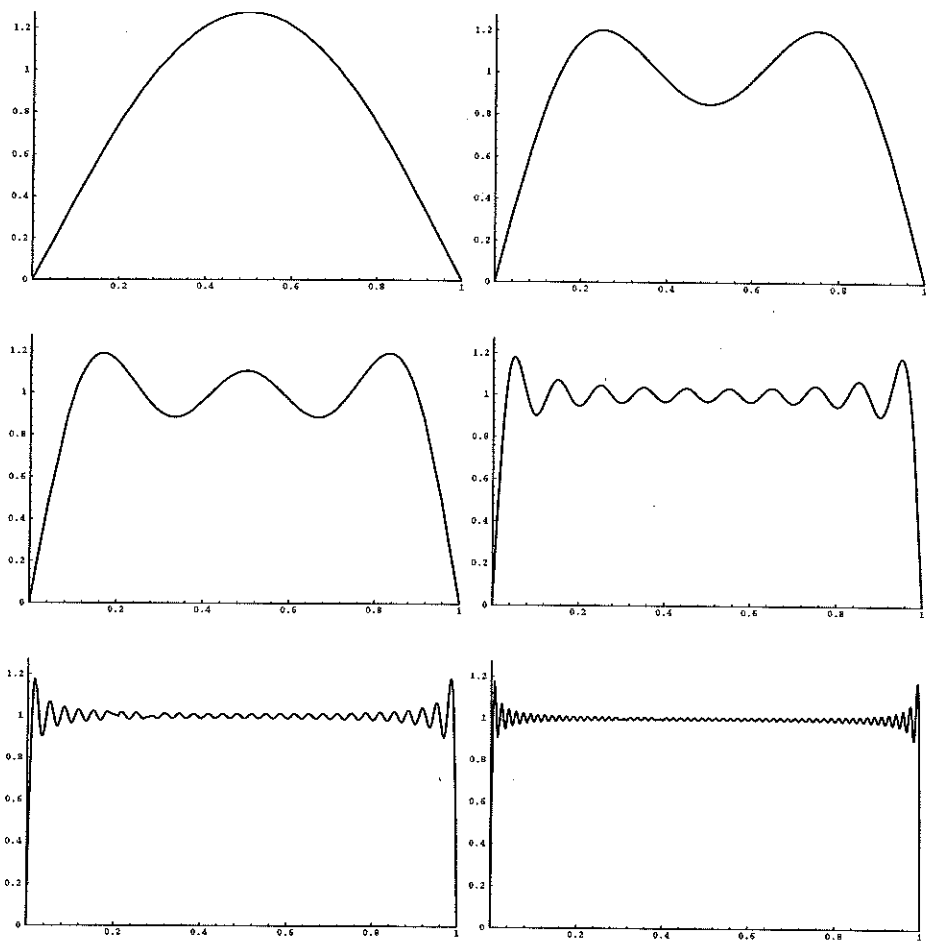
$$= \frac{4T_0}{\pi}\left[\sin\left(\frac{\pi x}{\ell}\right) e^{-\frac{D\pi^2 t}{\ell^2}} + \underbrace{\frac{1}{3}\sin\left(\frac{3\pi x}{\ell}\right) e^{-\frac{9D\pi^2}{\ell^2}t} + \dots}_{\text{higher terms die off quickly}}\right]$$



→ see relaxation.nb on Blackboard

```
partial[m_,t_]=4/Pi*Sum[Sin[(2n+1) Pi x]*Exp[-(2n+1)^2 Pi^2 t]/(2n+1), {n,0,m}]
Display["!psfix > sq60.eps", Plot[partial[59,0],{x,0,1},
PlotRange->{{0,1}, {0,4/Pi}}]]
```

Compared – square



← 18% overshoot

Figure 1: Fourier decomposition of square wave with 1, 2, 3, 10, 30, and 60 terms

relaxation.nb

```
In[7]:= amp = 4 / Pi;

In[8]:= mode[x_, t_, n_] = amp / n * Sin[n * Pi x] * Exp[- (Pi n) ^2 t];

In[9]:= partialsum[x_, t_, m_] := Sum[mode[x, t, n], {n, 1, m, 2}];

In[10]:= Animate[
  Plot[partialsum[x, t, 119], {x, 0, 1}, PlotRange → {{0, 1}, {0, 1.2}}], {t, 0, .5}]
```

↳ 60 terms

OBSOLETE

from composite-square.tex [after \end{document}]

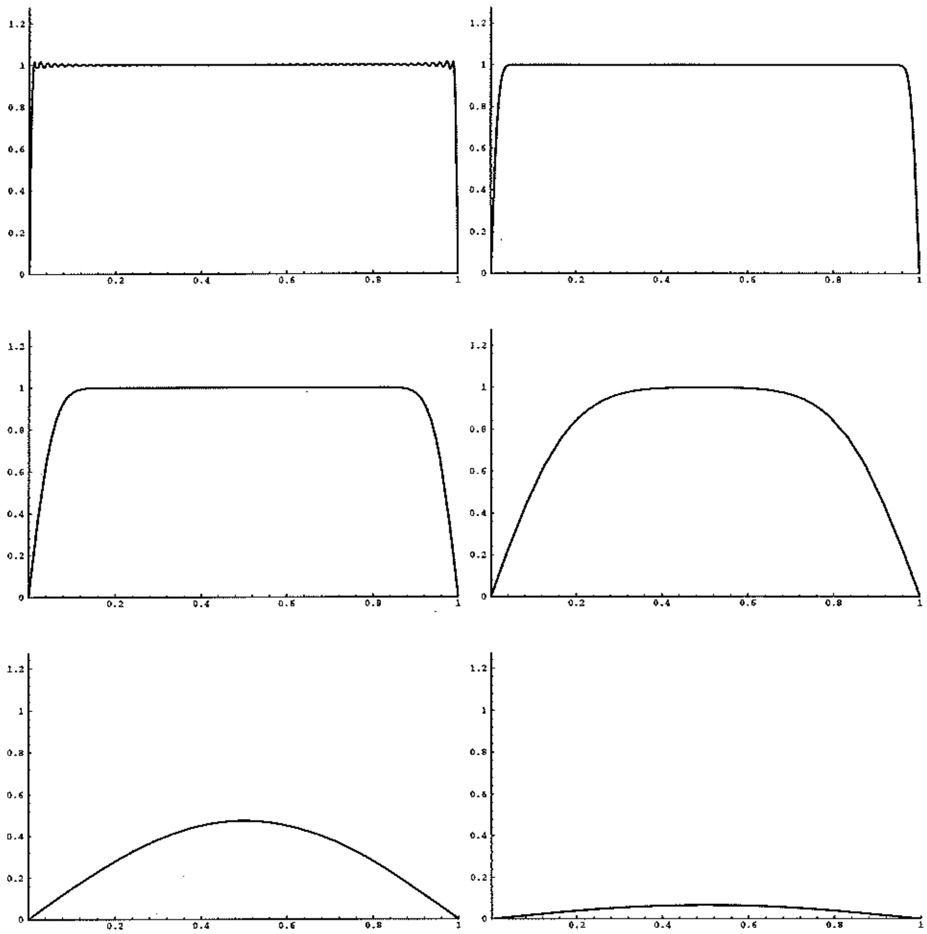
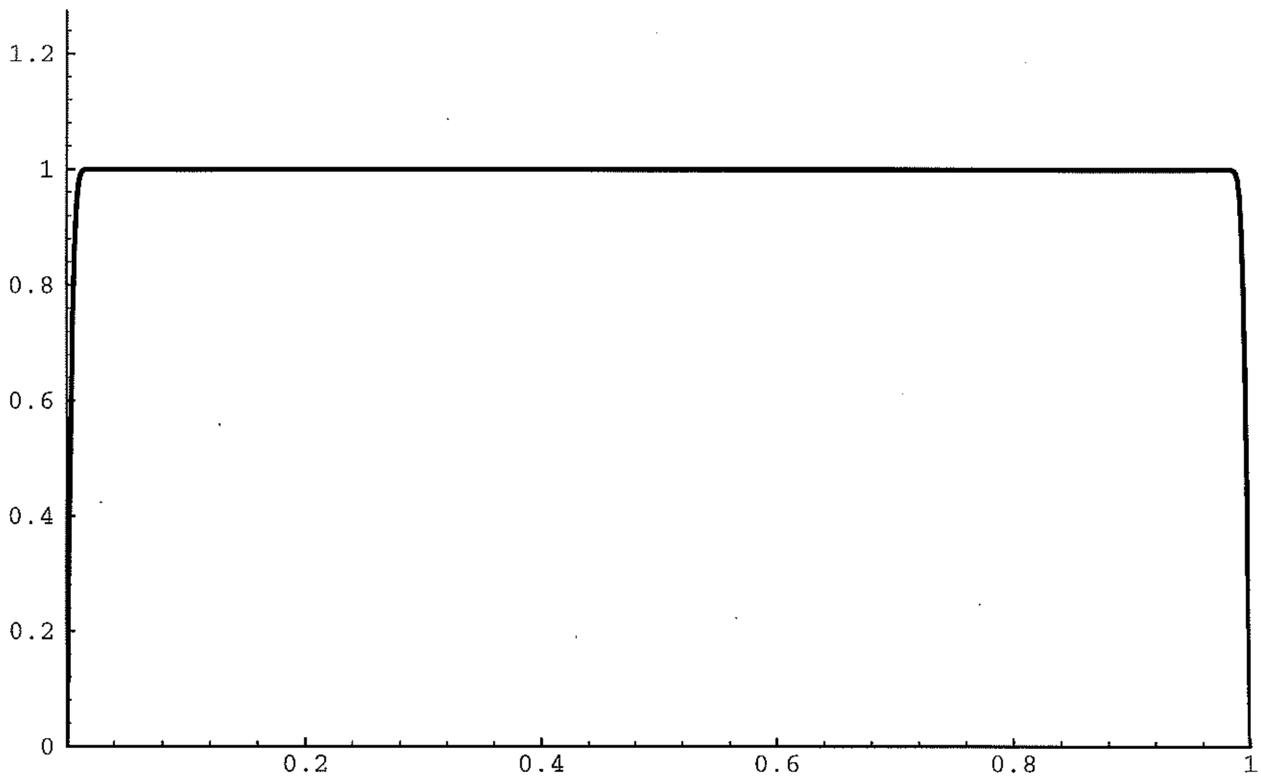


Figure 2: Thermal time evolution of square wave (with 60 terms) at $t = 10^{-5}$, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, and 0.3

$t = 0.00001$
w/ 160 terms in Fourier series



Other possible topics

✓ Now a problem

If one end of rod held at $T=0$ and other at $T=T_0$

find $T(x,t)$ given initial profile

Since steady-state soln is $T_0 \frac{x}{\ell}$

subtract this from initial profile & solve

[discussed in S'02]

Insulated ends: $\frac{\partial T}{\partial x}=0$ at $x=0$

use $\cos(\ )$

Neumann b.c.