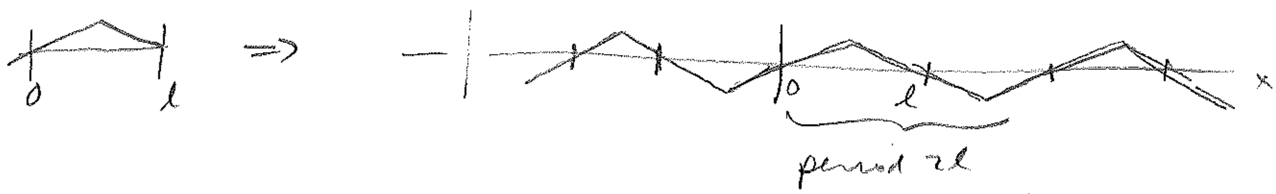
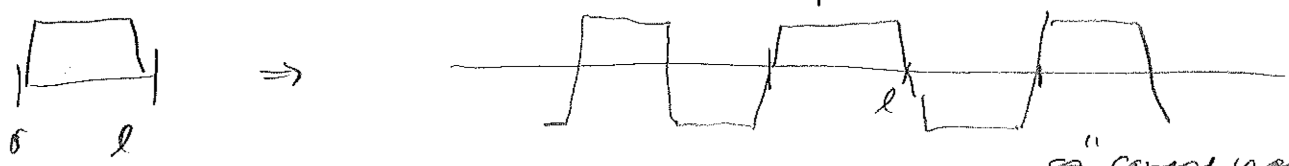
I skipped 2C in 2014 + thereafter

A fcn on a finite interval can be extended into a periodic fcn.




Any periodic fcn (which is finite & has a finite # of discontinuities) can be written as a sum of sines + cosines (= Fourier series)

Let $f(x)$ be periodic w/ period $2l$, ie $f(x+2l)=f(x)$

then

$$\boxed{f(x) = \underbrace{\frac{A_0}{2}}_{\text{"offset"}} + \sum_{n=1}^{\infty} \left[ A_n \cos(k_n x) + B_n \sin(k_n x) \right]}$$

where $A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(k_n x)\,dx$

$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(k_n x)\,dx$

If $f(x)$ is even, all $B_n = 0$

If $f(x)$ is odd, all $A_n = 0$

Rewrite in terms of complex exponentials

$$f(x) \equiv \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left[ A_n \left( \frac{e^{ik_n x} + e^{-ik_n x}}{2} \right) + B_n \left( \frac{e^{ik_n x} - e^{-ik_n x}}{2i} \right) \right]$$

$$= \frac{1}{2} \underbrace{A_0}_{C_0} + \sum_{n=1}^{\infty} \left[ \frac{1}{2} \underbrace{(A_n - iB_n)}_{\equiv C_n} e^{ik_n x} + \frac{1}{2} \underbrace{(A_n + iB_n)}_{C_n^*} e^{-ik_n x} \right]$$

$$= \frac{1}{2} C_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left[ C_n e^{ik_n x} \right]$$

use Fourier's trick to show

$$C_n = A_n - iB_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \left[ \cos(k_n x) - i \sin(k_n x) \right]$$

$$= \frac{1}{\ell} \int_{-\ell}^{\ell} f(x)\, e^{-ik_n x}\, dx$$

## Driven harmonic oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

Let $F(t)$ be an arbitrary periodic function of period $T$.

$\Rightarrow$ it can be written as a sum of sines + cosines of frequencies $\omega_n = \frac{2\pi n}{T}$

(N.B. $\omega_0 = \sqrt{\frac{k}{m}}$, natural frequency) not zero!

Step 1: write $F(t)$ as a Fourier series

$$F(t) = \frac{1}{2} c_0 + \sum_{n=1}^{\infty} Re\left( c_n e^{i\omega_n t} \right)$$

where $c_n = \frac{2}{T} \int_0^T F(t) e^{-i\omega_n t} dt$

Step 2: Steady-state response to a single harmonic mode $Re\left(c_n e^{i\omega_n t}\right)$ is

$$x(t) = Re\left[ \frac{c_n/m}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n} e^{i\omega_n t} \right]$$

Step 3: Superposition principle

$$x(t) = \frac{c_0}{2m\omega_0^2} + \sum_{n=1}^{\infty} Re\left[ \frac{c_n/m}{\omega_0^2 - \omega_n^2 + i\gamma\omega_n} e^{i\omega_n t} \right]$$

[Response is large if $\omega_n \approx \omega_0$ $\Rightarrow$ tuning a radio]