Alternative approach to solving the wave eqn

Consider a normal mode solution (standing wave)

$$\eta_n(x,t) = \sin(k_n x)\cos(\omega_n t), \quad \text{where } \omega_n = v k_n$$

$$= \frac{1}{2}\left[\sin(k_n x + \omega_n t) + \sin(k_n x - \omega_n t)\right]$$

$$= \frac{1}{2}\left[\sin(k_n[x+vt]) + \sin(k_n[x-vt])\right]$$

sine wave travelling to left w/ speed v

sine wave travelling to right w/ speed v

[Consider a node] $x - vt = 0$

$x_{node} = vt$

In fact, any shape whatsoever travelling to the left or to the right w/ speed v is a solution to the wave eqn

$$\frac{\partial^2 \eta}{\partial t^2} - v^2 \frac{\partial^2 \eta}{\partial x^2} = 0$$

Consider an arbitrary function $f_1(\xi)$

If $\xi = x - vt$, this represents a shape travelling to the right w/ speed v

Consider

$$\frac{\partial f_1}{\partial x} = \underset{\uparrow}{\frac{df_1}{d\xi}} \underbrace{\frac{\partial \xi}{\partial x}}_{1} = f_1' \qquad \text{where } f' \text{ denotes derivative w.r.t. argument of } f$$

N.B. ordinary derivative because f_1 only depends on one indep. variable

$$\frac{\partial^2 f_1}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{df_1}{d\xi}\right) = \frac{d}{d\xi}\left(\frac{df_1}{d\xi}\right)\frac{\partial \xi}{\partial x} = f_1''$$

$$\frac{\partial f_1}{\partial t} = \frac{df_1}{d\xi}\underbrace{\frac{\partial \xi}{\partial t}}_{-v} = -v f_1'$$

$$\frac{\partial^2 f_1}{\partial t^2} = \frac{d}{d\xi}\left(-v f_1'(\xi)\right)\underbrace{\frac{\partial \xi}{\partial x}}_{-v} = v^2 f_1''$$

$$\therefore \frac{\partial^2 f_1}{\partial t^2} - v^2 \frac{\partial^2 f_1}{\partial x^2} = v^2 f_1'' - v^2 f_1'' = 0 \quad \Rightarrow f_1(x-vt) \text{ solves the wave equation}$$

Similarly, consider another arbitrary function $f_2(\zeta)$ where $\zeta = x + vt$. This represents a shape travelling to the left w/ speed v

Analogous proof shows that $f_2(\zeta)$ obeys the wave equation

Linearity $\Rightarrow$ $\eta(x,t) = f_1(x - vt) + f_2(x + vt)$ (d'Alembert's solution)

$$\eta(x,t) = f_1(x-vt) + f_2(x+vt)$$

The functions f_1 + f_2 are determined by initial conditions

Suppose the string has initial profile given by $\eta_0(x)$

$$\eta(x,0) = f_1(x) + f_2(x) = \eta_0(x)$$

Suppose also that the string is not moving: $\frac{\partial \eta}{\partial t}(x,0) = 0$

$$\frac{\partial \eta}{\partial t} = -v f_1'(x-vt) + v f_2'(x+vt)$$

$$\frac{\partial \eta}{\partial t}(x,0) = -v f_1'(x) + v f_2'(x) = 0$$

$$\Rightarrow f_2'(x) = f_1'(x)$$

$$f_2(x) = f_1(x) + c$$

Then $f_1(x) + [f_1(x) + c] = \eta_0(x) \Rightarrow$

$$f_1(x) = \tfrac{1}{2}[\eta_0(x) - c]$$

$$f_2(x) = \tfrac{1}{2}[\eta_0(x) - c]$$

Then the solution of the wave equation is

$$\eta(x,t) = \tfrac{1}{2}[\eta_0(x-vt) + \eta_0(x+vt)]$$

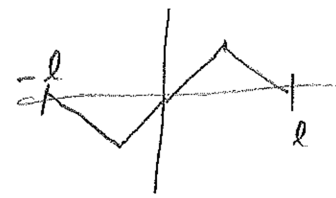
We are only given $\eta_0(x)$ for $0 < x < l$, e.g. 

How determine $\eta_0(x)$ outside this range? Boundary conditions

$\eta(0,t) = 0$

But $\eta(0,t) = \frac{1}{2}[\eta_0(-vt) + \eta_0(vt)]$

$\Rightarrow \eta_0(-vt) = -\eta_0(vt) \quad \Rightarrow \eta_0$ is an odd function



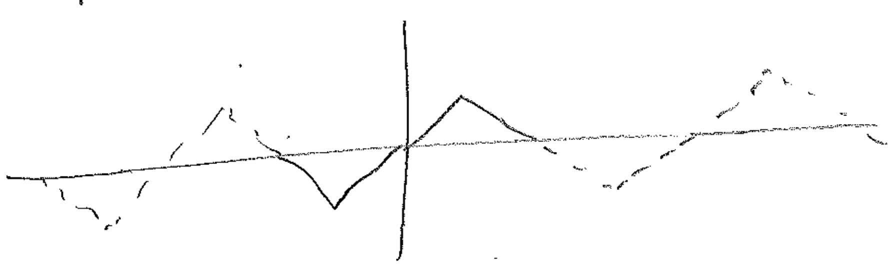
$\eta(l,t) = 0$

But $\eta(l,t) = \frac{1}{2}[\eta_0(l-vt) + \eta_0(l+vt)]$

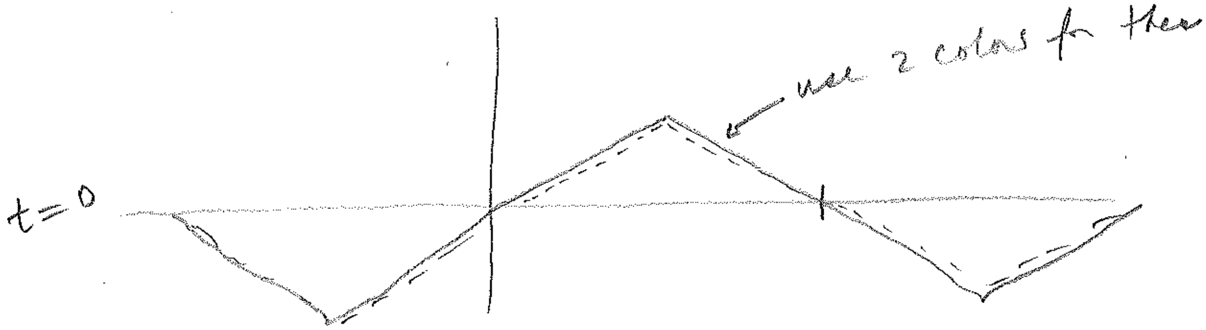
odd, so $= -\eta_0(-l+vt)$

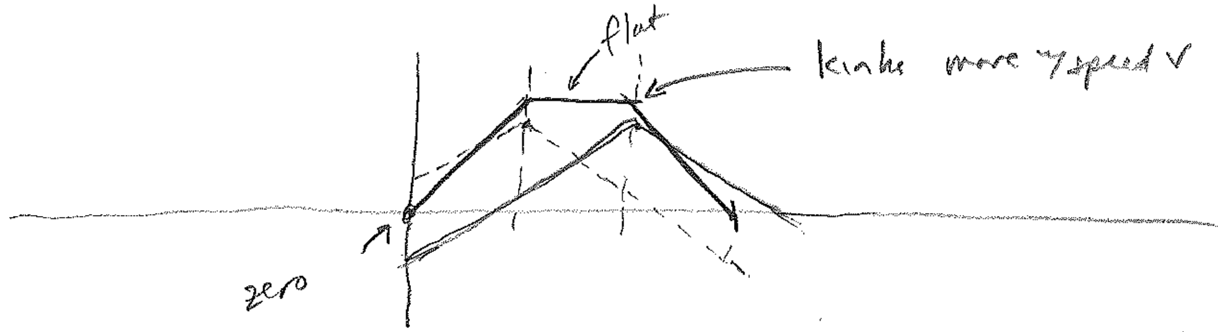
$\Rightarrow \eta_0(l+vt) = \eta_0(-l+vt)$

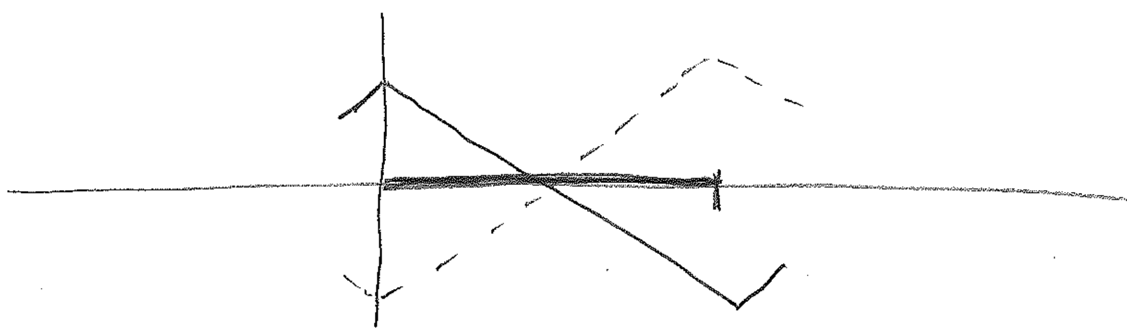
$\Rightarrow \eta_0$ is periodic with period $2l$



$$\eta(x,t) = \frac{1}{2}\left[\eta_0(x-vt) + \eta_0(x+vt)\right]$$

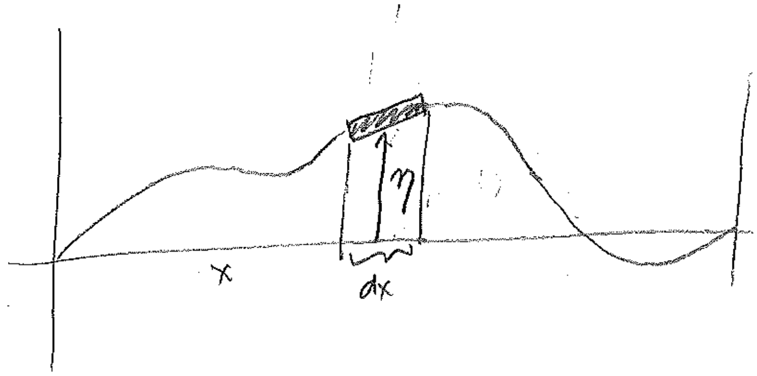






etc.

Energy of a string



Kinetic energy of infinitesimal element

$$dK = \frac{1}{2}(dm)\,\dot{\eta}^2 = \frac{1}{2}\rho\, dx\, \dot{\eta}^2 \qquad \rho = \text{linear density}$$

$$K = \int_0^{l} \frac{1}{2}\rho\, \dot{\eta}^2\, dx$$

Potential energy $U = T \underbrace{\Delta l}_{\text{change in total length}}$ (assume const tension T for Δl small)

Length of infinitesimal segment $dl = \sqrt{dx^2 + d\eta^2} = dx\sqrt{1+(\eta')^2}$

$$U = T\int_0^{l}\left[\sqrt{1+\eta'^2} - 1\right]dx \approx \int_0^{l} \frac{1}{2}T(\eta')^2\,dx$$

For simplicity, assume string initially at rest ($\Rightarrow b_n = 0$)

$$\eta(x,t) = \sum_{n=1}^{\infty} a_n \cos(\omega_n t)\sin(k_n x)$$

$$\dot{\eta} = -\sum_{n=1}^{\infty} \omega_n a_n \sin(\omega_n t)\sin(k_n x)$$

$$\dot{\eta}^2 = \left[\sum_{n=1}^{\infty} \omega_n a_n \sin(\omega_n t)\sin(k_n x)\right]\left[\sum_{m=1}^{\infty} \omega_m a_m \sin(\omega_m t)\sin(k_m t)\right]$$

diagonal terms are those w/ $n = m$
off-diagonal terms " $n \neq m$

$$K = \frac{1}{2}\rho \int_0^{l} \dot{\eta}^2\, dx$$

$$= \frac{1}{2}\rho\left[\sum_{n=1}^{\infty} \omega_n^2 a_n^2 \sin^2(\omega_n t)\int_0^{l} \sin^2(k_n x)\, dx + \sum_{n=1}^{\infty}\sum_{\substack{m=1 \\ (m\neq n)}}^{\infty} \omega_n a_n \sin(\omega_n t)\, \omega_m a_m \sin(\omega_m t)\int_0^{l} \sin(k_n x)\sin(k_m x)\, dx\right]$$

Integrate each term over x.
Off-diagonal terms vanish: $\int_0^{l} \sin(k_n x)\sin(k_m x)\, dx = 0$, $n \neq m$
Diagonal terms give: $\int_0^{l} \sin^2(k_n x)\, dx = \frac{1}{2} l$

$$K = \sum_{n=1}^{\infty} \frac{1}{2}\rho\, \omega_n^2 a_n^2 \sin^2(\omega_n t)\, \frac{l}{2}$$

Observe: $K = 0$ if $t = 0$ (consistent w/ initial condition)

A similar calculation for U gives

$$U = \sum_{n=1}^{\infty} \frac{1}{2} \underbrace{T k_n^2}_{\rho\omega_n^2} a_n^2 \cos^2(\omega_n t) \frac{l}{2}$$

Recall $v^2 \equiv \dfrac{T}{\rho}$ so $T k_n^2 = \rho v^2 k_n^2 = \rho \omega_n^2$

$\therefore$ total energy is

$$E = K + V = \sum_{n=1}^{\infty} \frac{1}{4} \rho l\, \omega_n^2 a_n^2 \left[\underbrace{\sin^2(\omega_n t) + \cos^2(\omega_n t)}_{1} \right]$$

$$= \frac{1}{4} \rho l \sum_{n=1}^{\infty} \omega_n^2 a_n^2$$

If string is <u>also</u> moving initially, then

$$E = \frac{1}{4} \rho l \sum_{n=1}^{\infty} \omega_n^2 (a_n^2 + b_n^2)$$