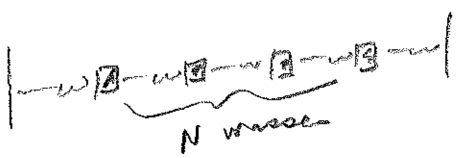
Ordinary differential equations (one independent variable)

1) first order
   a) separable
   b) linear: homog & inhomog

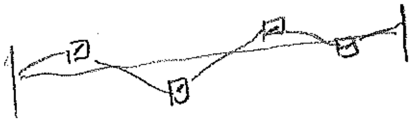
2) second order
   a) linear
   b) systems of equations

partial differential equations (2 or more indep variables)



N degrees of freedom
$\Rightarrow$ N normal modes

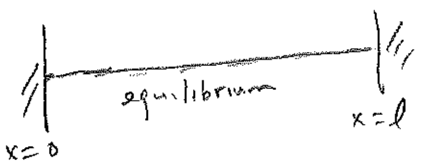
Let $N \to \infty$: elastic medium $\to$ wave eqn
$\Rightarrow$ longitudinal waves (slinky, sound)



$N \to \infty$: wave eqn
$\Rightarrow$ transverse waves (string; EM waves)

[B 13-47]

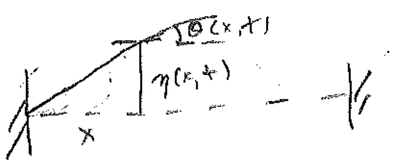
Waves on a string



l = length of string

T = tension $\approx$ const

$\rho = \frac{\text{mass}}{\text{length}}$



$\eta(x,t)$ = transverse displacement from equilibrium

$\theta(x,t)$ = angle string makes wrt horizontal

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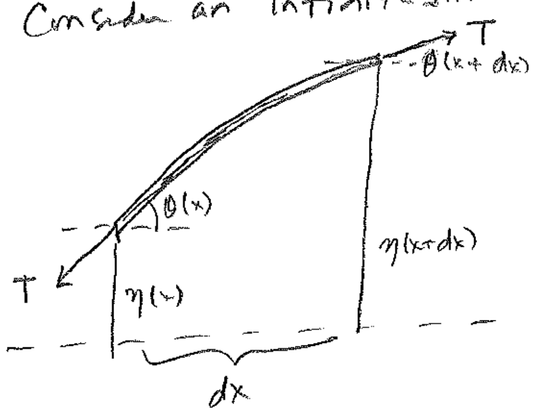
slope of string $\frac{\partial \eta}{\partial x} = \tan\theta$

Assume slope is small all along the string $\begin{cases} \tan\theta = \theta - \frac{1}{3}\theta^3 + \dots \approx \theta \\ \cos\theta = 1 - \frac{1}{2}\theta^2 + \dots \approx 1 \end{cases}$

(small ∡ approx)

Consider an infinitesimal segment



Net force on element

$dF_x = T\cos\theta(x+dx) - T\cos\theta(x)$

$dF_y = T\sin\theta(x+dx) - T\sin\theta(x)$

$\cos\theta \approx 1 \Rightarrow dF_x = T - T = 0$

$\sin\theta \approx \theta \approx \tan\theta \approx \frac{\partial \eta}{\partial x}$

$$dF_y = T \underbrace{\frac{\partial \eta}{\partial x}(x+dx)}_{\frac{\partial \eta}{\partial x}(x) + \frac{\partial^2 \eta}{\partial x^2}dx} - T\frac{\partial \eta}{\partial x}(x) = T\frac{\partial^2 \eta}{\partial x^2}dx$$

net force proportional to curvature.

Net force on the segment causes it to accelerate

$dF_y = (dm)\, a_y$ $dm = \rho\, dx$

$a_y = \frac{\partial^2 \eta}{\partial t^2}$

$T \frac{\partial^2 \eta}{\partial x^2} dx = \rho \frac{\partial^2 \eta}{\partial t^2} dx$

$\frac{\partial^2 \eta}{\partial t^2} - \left(\frac{T}{\rho}\right) \frac{\partial^2 \eta}{\partial x^2} = 0$

T : units of force $\frac{kg \cdot m}{s^2}$

ρ : units of density $\frac{kg}{m}$

$\frac{T}{\rho}$: units of $\frac{m^2}{s^2} = (\text{velocity})^2$

Define $v^2 = \frac{T}{\rho}$

$\boxed{\frac{\partial^2 \eta}{\partial t^2} - v^2 \frac{\partial^2 \eta}{\partial x^2} = 0}$ wave equation (2nd order linear PDE)

Expect string to vibrate → try normal mode ansatz

$\eta(x,t) = \underset{\substack{\uparrow \\ \text{complex constant}}}{c}\ \underset{\substack{\uparrow \\ \text{real function}}}{f(x)}\ e^{i\omega t}$

Special case of separable ansatz $\eta(x,t) = f(x)\, g(t)$

$\frac{\partial^2 \eta}{\partial t^2} = -\omega^2 c f(x) e^{i\omega t}$

$v^2 \frac{\partial^2 \eta}{\partial x^2} = c \frac{d^2 f}{dx^2} e^{i\omega t}$

$\Rightarrow \frac{d^2 f}{dx^2} + \left(\frac{\omega^2}{v^2}\right) f = 0$

Define $k = \frac{\omega}{v}$ = wave number [we'll see why later]

$\boxed{\frac{d^2 f}{dx^2} + k^2 f = 0}$

Same form as harmonic oscillator $\left(\frac{d^2 x}{dt^2} + \omega^2 x = 0 \Rightarrow x = A\cos(\omega t) + B\sin(\omega t)\right)$

$f(x) = A\cos(kx) + B\sin(kx)$

[now we see why called wave number: wiggles per unit length]

A, B are real because f is real

PDE requires both initial conditions + boundary condition

Dirichlet b.c. $\rightarrow$ constraints on $f(x)$
Neumann b.c $\rightarrow$ constraints on $\frac{df}{dx}$

Here we have Dirichlet conditions: string anchored at both ends
$f(0) = f(\ell) = 0$

$f(x) = A\cos kx + B\sin kx$

$f(0) = A \quad \Rightarrow \quad A = 0$

$f(\ell) = B\sin(k\ell) \quad \Rightarrow \quad B = 0 \quad \text{or} \quad \sin(k\ell) = 0 \Rightarrow k\ell = \pi n$
$n = \text{integer}$

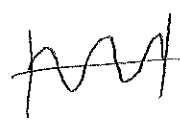
Consider $n = 1 \Rightarrow k_1 = \frac{\pi}{\ell}$

$f_1(x) = B_1 \sin\left(\frac{\pi x}{\ell}\right)$

$\omega_1 = v k_1 = \sqrt{\frac{T}{\rho}}\,\frac{\pi}{\ell}$ = fundamental frequency

$n > 1$ are called overtones or harmonics

$k_n = \frac{\pi n}{\ell} = n k_1$



$f_n(x) = B_n \sin\left(\frac{\pi n x}{\ell}\right)$

$\omega_n = v k_n = n\omega_1$

---

nth Normal mode solution (a.k.a. Fourier mode)

$$\eta_n(x,t) = \underbrace{c\,B_n}_{\text{call this } c_n \text{ (complex constant)}} \sin(k_n x)\, e^{i\omega_n t}$$

general solution is a linear combination (take real part)

$$\eta(x,t) = \sum_{n=1}^{\infty} \mathrm{Re}\left[c_n e^{i\omega_n t}\right] \sin(k_n x)$$

Let $c_n = a_n - i b_n$

$$\eta(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right] \sin(k_n x)$$

Bernoulli solution

The undetermined coefficients a_n, b_n in the general solution are determined by initial conditions $\eta(x,0)$ and $\dot{\eta}(x,0)$

Set $t=0$ in general solution

$$\eta(x,0) = \sum_{n=1}^{\infty} a_n \sin(k_n x) \qquad \text{recall } k_n = \frac{\pi}{\ell} n$$

$$\dot{\eta}(x,0) = \sum_{n=1}^{\infty} b_n \omega_n \sin(k_n x)$$

To isolate a_n & b_n, use "Fourier's trick":
multiply both sides by $\sin(k_j x)$ where j = positive integer
then integrate from 0 to ℓ.

[not particularly easier using $e^{i\theta}$]

only one term survives

$$\int_0^\ell \eta(x,0) \sin(k_j x)\, dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^\ell \sin(k_j x) \sin(k_n x)\, dx} = a_j \frac{\ell}{2}$$

("orthogonal")
- if $n \neq j$, this vanishes
- if $n = j$, this equals $\frac{\ell}{2}$ $\left(\langle \sin^2 \rangle = \frac{1}{2}\right)$

(Proof to follow)

Therefore

$$\boxed{\begin{aligned} a_j &= \frac{2}{\ell} \int_0^\ell \eta(x,0) \sin(k_j x)\, dx \\ b_j &= \frac{2}{\ell \omega_j} \int_0^\ell \dot{\eta}(x,0) \sin(k_j x)\, dx \end{aligned}}$$

Proof of assertion: $\sin(k_j x)\sin(k_n x) = \frac{1}{2}\left[\cos(k_j - k_n)x + \cos(k_j + k_n)x\right]$

If $n \neq j$ then integrates to $\frac{1}{2}\left[\frac{\sin(k_j - k_n)x}{k_j - k_n} + \frac{\sin(k_j + k_n)x}{k_j + k_n}\right]\Big|_0^\ell = 0$

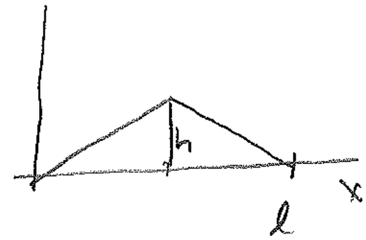
because $\sin(k_j - k_n)\ell = \sin(\pi(j-n)) = 0$

If $n = j$, then $\sin^2(k_j x) = \frac{1}{2}\left[1 + \cos(2k_j x)\right]$

This integrates to $\frac{1}{2}\left[x + \frac{\sin(2k_j x)}{2k_j}\right]\Big|_0^\ell = \frac{\ell}{2}$

Example: string is displaced by h at midpoint, then released

Initial profile given by a function $\eta_0(x)$



$$\eta_0(x) = \begin{cases} \frac{2h}{l} x & 0 < x < \frac{l}{2} \\ 2h - \left(\frac{2h}{l}\right) x & \frac{l}{2} < x < l \end{cases}$$

Initial velocity is zero, $\dot{\eta}(x,0)$ $\Rightarrow$ $b_j = 0$ for all n

$$a_j = \frac{2}{l} \int_0^l \eta_0(x) \sin(k_j x)\, dx$$

$$= \frac{2}{l} \left[\int_0^{\frac{l}{2}} \frac{2h}{l} x \sin(k_j x)\, dx + \int_{\frac{l}{2}}^{l} \left(2h - \frac{2hx}{l}\right) \sin(k_j x)\, dx \right]$$

How compute these integrals? IBP!

But let's derive a more general form for a_j using IBP

Product rule $\quad d(uv) = u\,dv - v\,du$

$$u\,dv = d(uv) - v\,du$$

$$\int u\,dv = uv\Big| - \int v\,du \qquad \text{IBP}$$

$$a_j = \frac{2}{\ell}\int_0^{\ell} \eta_0(x)\,\sin(k_j x)\,dx$$

Let $\quad u = \eta_0(x) \qquad dv = \sin(k_j x)\,dx$

$\quad du = \eta_0'(x)\,dx \qquad v = -\frac{1}{k_j}\cos(k_j x)$

$$a_j = \frac{2}{\ell}\Big[\underbrace{-\frac{1}{k_j}\eta_0(x)\cos(k_j x)\Big|_0^{\ell}}_{\substack{\text{vanishes for Dirichlet b.c.}\\ \text{because } \eta_0(0)=\eta_0(\ell)=0}} + \frac{1}{k_j}\int_0^{\ell}\eta_0'(x)\cos(k_j x)\,dx\Big]$$

Caveat: be <u>careful</u> with this expression if $\eta_0(x)$ is <u>not continuous</u> (then $\eta_0'(x) = \infty$)

Recall $k_j = \frac{\pi}{\ell} j \;\Rightarrow\; a_j = \frac{2}{\pi j}\int_0^{\ell}\eta_0'(x)\cos\left(\frac{\pi j}{\ell}x\right)dx$

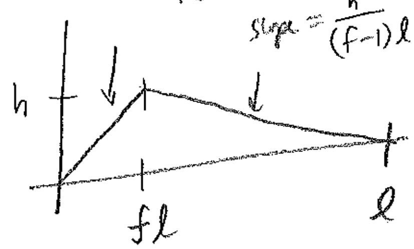
For our problem: $\eta_0'(x) = \begin{cases} \frac{2h}{\ell} & 0 < x < \frac{\ell}{2} \\ -\frac{2h}{\ell} & \frac{\ell}{2} < x < \ell \end{cases}$

$$a_j = \frac{2}{\pi j}\Big[\underbrace{\int_0^{\frac{\ell}{2}} \frac{2h}{\ell}\cos\left(\frac{\pi j}{\ell}x\right)dx}_{\frac{2h}{\pi j}\sin\left(\frac{\pi j}{\ell}x\right)\Big|_0^{\ell/2}} + \underbrace{\int_{\frac{\ell}{2}}^{\ell}\left(-\frac{2h}{\ell}\right)\cos\left(\frac{\pi j}{\ell}x\right)dx}_{-\frac{2h}{\pi j}\sin\left(\frac{\pi j}{\ell}x\right)\Big|_{\frac{\ell}{2}}^{\ell}}\Big]$$

$$= \frac{2}{\pi j}\left(\frac{2h}{\pi j}\right)\left[\sin\left(\frac{\pi j}{2}\right) - 0 - \sin(\pi j) + \sin\left(\frac{\pi j}{2}\right)\right]$$

$$= \frac{8h}{\pi^2 j^2}\sin\left(\frac{\pi j}{2}\right)$$

Nice trick (Varun Wadia's dad)



$$a_m = \frac{2}{\ell}\int_0^\ell \eta(x,0)\,\underbrace{\sin\left(\frac{m\pi x}{\ell}\right)}_{-\frac{\ell}{m\pi}\frac{d}{dx}\cos\left(\frac{m\pi x}{\ell}\right)}dx$$

$$= \frac{2}{m\pi}\int_0^\ell \eta'(x,0)\cos\left(\frac{m\pi x}{\ell}\right)dx$$

$$= \frac{2}{m\pi}\left[\frac{h}{f\ell}\int_0^{f\ell}\cos\left(\frac{m\pi x}{\ell}\right)dx - \frac{h}{(1-f)\ell}\int_{f\ell}^{\ell}\cos\left(\frac{m\pi x}{\ell}\right)dx\right]$$

$$= \frac{2}{m\pi}\left[\frac{h}{m\pi f}\sin(m\pi f) - \frac{h}{(1-f)m\pi}\left[\underbrace{\sin(m\pi)}_{0} - \sin(m\pi f)\right]\right]$$

$$= \frac{2h\,\sin(m\pi f)}{(m\pi)^2 f(1-f)}$$

e.g. $f = \frac{1}{2} \Rightarrow \frac{8h}{(m\pi)^2}\sin\left(\frac{m\pi}{2}\right)$

$f = \frac{1}{3} \Rightarrow \frac{9h}{(m\pi)^2}\sin\left(\frac{m\pi}{3}\right)$

$f = \frac{2}{3} \Rightarrow \frac{9h}{(m\pi)^2}\sin\left(\frac{2\pi m}{3}\right)$

(Review)

$$\frac{\partial^2\eta}{\partial t^2} - v^2\frac{\partial^2\eta}{\partial x^2} = 0, \qquad \text{boundary conditions: } \eta(0,t) = \eta(\ell,t) = 0$$

normal mode solutions $\eta_n(x,t) = c_n e^{i\omega_n t}\sin(k_n x)$ $\qquad k_n = \frac{\pi n}{\ell}$, $\omega_n = v k_n$

"standing waves"

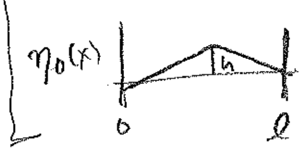
general solution (linearity) $\quad \eta(x,t) = \sum_{n=1}^{\infty}(a_n\cos\omega_n t + b_n\sin\omega_n t)\sin k_n x$

initial conditions: $\eta(x,0)$ and $\frac{\partial\eta}{\partial t}(x,0) = \dot\eta(x,0)$

$$\eta(x,0) = \sum a_n \sin k_n x \quad \Longrightarrow \quad a_n = \frac{2}{\ell}\int_0^\ell \eta(x,0)\sin(k_n x)\,dx$$

$$\dot\eta(x,0) = \sum b_n\omega_n \sin k_n x \quad \Longrightarrow \quad b_n = \frac{2}{\ell\omega_n}\int_0^\ell \dot\eta(x,0)\sin(k_n x)\,dx$$

string displaced at midpoint, then released



$$\Longrightarrow \quad b_n = 0, \qquad a_n = \frac{8h}{\pi^2}\,\frac{\sin\left(\frac{\pi n}{2}\right)}{n^2}$$

$$\eta(x,0) = \frac{8h}{\pi^2}\left[\sin(k_1 x) - \frac{1}{9}\sin(3k_1 x) + \frac{1}{25}\sin(5k_1 x) + \cdots\right]$$

[show graphs: Blackboard: composite.pdf]

Then:

$$\eta(x,t) = \frac{8h}{\pi^2}\left[\sin(k_1 x)\cos(\omega_1 t) - \frac{1}{9}\sin(3k_1 x)\cos(3\omega_1 t) + \frac{1}{25}\sin(5k_1 x)\cos(5\omega_1 t)\right]$$

[show animation: Blackboard. plucked.nb]

```
h=1
omega = 1
initial     =  h - Abs[ 2h(x-.5)]
partial[m_] = 8h/Pi^2 *Sum[(-1)^n Sin[(2n+1) Pi x] *
              Cos[(2n+1) omega t]/(2n+1)^2, {n,0,m}]
```

Composite_triangle.pdf

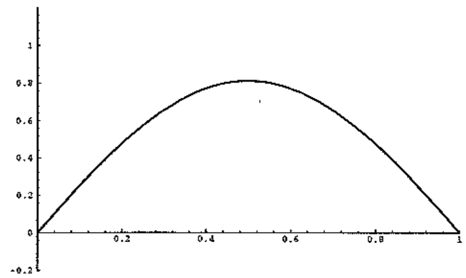


Figure 1: First term in Fourier series

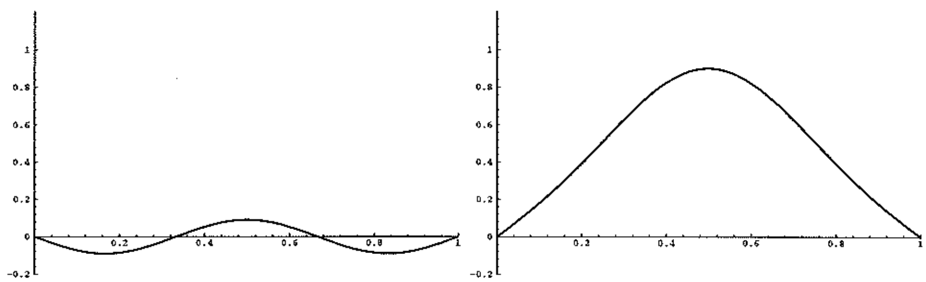


Figure 2: Second term, and sum of first two terms in Fourier series

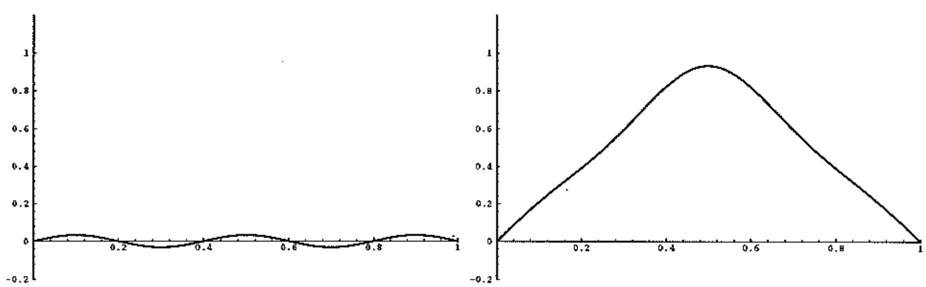


Figure 3: Third term, and sum of first three terms in Fourier series

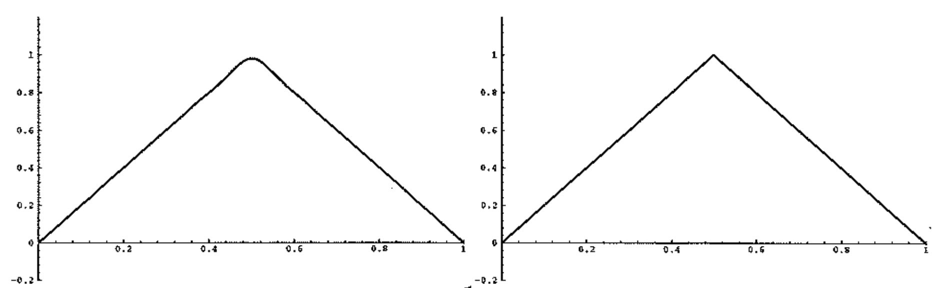


Figure 4: Sum of first 10 terms, and sum of first 100 terms

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OBSOLETE

This is after \end{document}

because of plucked.nb

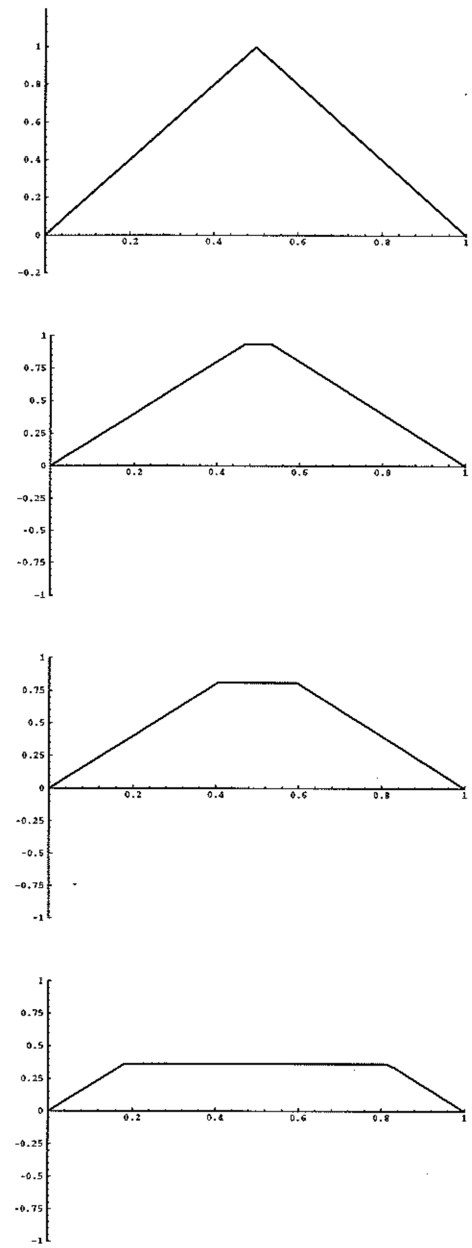

Figure 5: Time evolution of plucked string at $t = 0,\ 0.1,\ 0.3,\ 1.0$

plucked.nb

```
h = 1;
f = 1 / 2;
omega = 1;
amp = 2 h / (f * (1 - f) * Pi^2) ;

mode[x_, t_, n_] = amp * Sin[n Pi f] * Sin[n Pi x] * Cos[n omega t] / n^2;

partialsum[x_, t_, m_] := Sum[mode[x, t, n], {n, 1, m}];

Animate[ Plot[partialsum[x, t, 50],
  {x, 0, 1}, PlotRange → {{0, 1}, {-1.1, 1.1}}], {t, 0, 2 * Pi}]
```