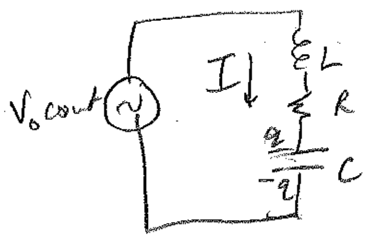
2014

mechanical damped driven oscillator $\quad m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t$

[They've seen RLC circuits in the problems]

RLC circuit w/ AC driving voltage



Let q = charge on capacitor (upper plate)

$$I = \frac{dq}{dt}$$

$$V_L + V_R + V_C = V_0 \cos\omega t$$

$$L\frac{dI}{dt} + RI + \frac{q}{C} = V_0 \cos\omega t$$

$$L\ddot{q} + R\dot{q} + \left(\frac{1}{C}\right)q = V_0 \cos\omega t$$

<u>Analogy</u>

<u>mech</u>	<u>electrical</u>
x	q
v	I
m	L
b	R
k	$1/C$
$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
$\gamma = \frac{b}{m}$	$\gamma = \frac{R}{L}$

$$Q = \frac{\omega_0}{\gamma}$$

complexify: $\quad L\ddot{q} + R\dot{q} + \frac{1}{C}q = V_0 e^{i\omega t}$.

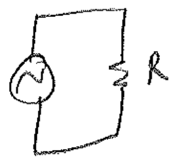
- Impedance Z is the response of the current to an AC voltage

$$I = \frac{V_0 e^{i\omega t}}{Z}$$ [then take real part]

$\leftarrow$ this eqn defines Z

$|Z|$ describes the "resistance" and the phase of Z describes how much the current leads or lags the driving voltage

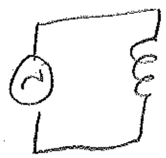
① resistor



$$V_R = IR = V_0 e^{i\omega t}$$

$$I = \frac{V_0 e^{i\omega t}}{R} \Rightarrow Z_R = R$$

② inductor



$$V_L = L\frac{dI}{dt} \quad V_0 e^{i\omega t}$$

$$\frac{dI}{dt} = \frac{V_0}{L} e^{i\omega t}$$

$$I = \frac{V_0}{i\omega L} e^{i\omega t} \Rightarrow Z_L = i\omega L$$

$$= \frac{V_0}{\omega L} e^{i(\omega t - \frac{\pi}{2})}$$

$\uparrow$ current lags voltage by 90°

③ capacitor

$$V_C = q/C = V_0 e^{i\omega t}$$

$$q = CV_0 e^{i\omega t}$$

$$I = i\omega C V_0 e^{i\omega t} = \frac{V_0 e^{i\omega t}}{\left(+\frac{1}{i\omega C}\right)} \Rightarrow Z_C = \frac{1}{i\omega C}$$

$$= \omega C V_0 e^{i(\omega t + \frac{\pi}{2})}$$

$\nwarrow$ current leads voltage by 90°

Impedances of elements in series add $Z = Z_1 + Z_2$

" " " parallel add inversely $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$

RLC circuit: $Z = Z_R + Z_L + Z_C = R + i\omega L - \frac{i}{\omega C}$

Mechanical impedance Z characterizes the response of the steady state velocity to a harmonic driving force

$$\dot{x} = \frac{F_0 e^{i\omega t}}{Z}$$ [N.B. don't include Re here]

Recall $m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}$

$$x = Ce^{i\omega t} \quad \text{(steady state)}$$

$$\Rightarrow C = \frac{F_0}{-m\omega^2 + ib\omega + k} = Ae^{i\phi}$$

where
$$\begin{cases} A = \dfrac{F_0/m}{\sqrt{(\omega^2-\omega_0^2)^2 + (\gamma\omega)^2}} \\ \phi = -\arctan\left(\dfrac{\gamma\omega}{\omega_0^2-\omega^2}\right) \end{cases}$$

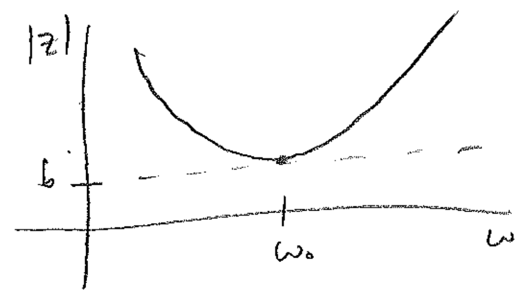
$$\dot{x} = i\omega Ce^{i\omega t} = e^{\frac{i\pi}{2}}\omega(Ae^{i\phi})e^{i\omega t} = \omega A \underbrace{e^{i(\omega t+\phi+\frac{\pi}{2})}}_{\text{hold onto this for a minute}}$$

$$\dot{x} = \frac{F_0}{(im\omega + b - i\frac{k}{\omega})} e^{i\omega t}$$

$$\Rightarrow Z = b + im\omega - \frac{ik}{\omega}$$ (check analogy w/ electrical)

$$= b\left[1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

Let $Z = |Z| e^{-i\phi_V}$



$$|Z| = b\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$$

Minimized at $\omega = \omega_0$

$$\phi_V = -\arctan\left[Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

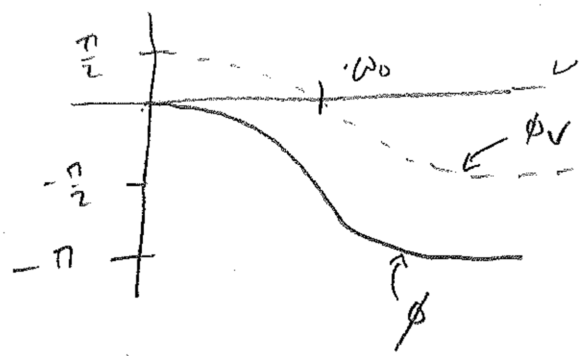
velocity phase

Then

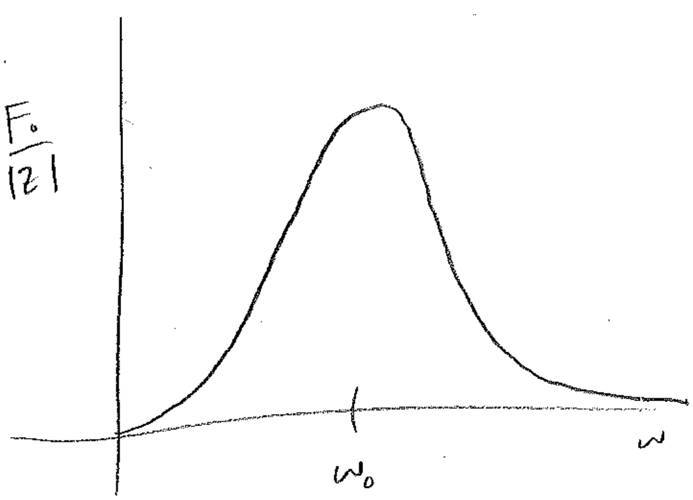
$$\dot{x} = \frac{F_0}{Z} e^{i\omega t} = \frac{F_0}{|Z|} e^{i(\omega t + \phi_V)}$$

so $\phi_V = \phi + \frac{\pi}{2}$

$\phi_V = 0$ at $\omega = \omega_0$



amplitude of velocity $= \omega A = \frac{F_0}{|Z|}$



Power dissipated by a damped driven oscillator (steady state)

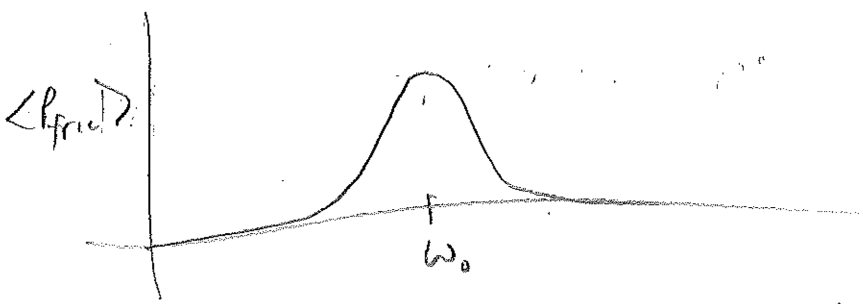
work done by friction $dW = \vec{F}_{fric} \cdot d\vec{r}$

Power dissipated by friction $P = \frac{dW}{dt} = \vec{F}_{fric} \cdot \vec{v} = (-b\dot{x})\dot{x} = -b\dot{x}^2$

Steady state $\dot{x} = \frac{F_0}{|Z|} \cos(\omega t + \phi_v)$

$$P_{fric} = -b \frac{F_0^2}{|Z|^2} \cos^2(\omega t + \phi_v)$$

$$\langle P_{fric} \rangle = -\frac{b}{2} \frac{F_0^2}{|Z|^2}$$



max power dissipated at $\omega = \omega_0$, where $|Z| = b$

$$\Rightarrow \langle P_{fric} \rangle_{max} = -\frac{F_0^2}{2b}$$

[the lighter the damping, the more power dissipated]

Driven RLC circuit: $\langle P_{fric} \rangle = -\frac{R V_0^2}{2|Z|^2}$

$$\langle P_{fric} \rangle_{max} = -\frac{V_0^2}{2R}$$