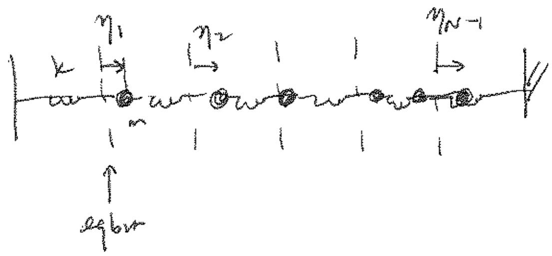
extra

~~JJ~~

JJ1

only done in 2002

Chain of $N-1$ masses in a line



η_n = displacement from eqbm position

$$m\ddot{\eta}_n = -k(\eta_n - \eta_{n-1}) - k(\eta_n - \eta_{n+1}) \qquad n = 1, \dots, N-1$$

where $\eta_0 = 0$, $\eta_N = 0$

$$\ddot{\eta}_n = -\omega_0^2\left[2\eta_n - \eta_{n+1} - \eta_{n-1}\right] \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

Normal mode: $\eta_n = f_n e^{i\omega t}$

$$(2\omega_0^2 - \omega^2) f_n = \omega_0^2 (f_{n+1} + f_{n-1}) \qquad \Rightarrow \text{det}\dots$$

$$\underbrace{\frac{f_{n+1} + f_{n-1}}{f_n}}_{\text{must be indep of } n.} = 2 - \left(\frac{\omega}{\omega_0}\right)^2$$

$$f_n = e^{in\theta} \Rightarrow 2\cos\theta = 2 - \left(\frac{\omega}{\omega_0}\right)^2 \Rightarrow \left(\frac{\omega}{\omega_0}\right)^2 = 2(1-\cos\theta)$$
$$= 2\left(2\sin^2\frac{\theta}{2}\right)$$

$$f_n = \text{Im}\, e^{in\theta}, \quad \text{Im}\left(e^{i(n\theta+\theta)} + e^{i(n\theta-\theta)}\right) = \text{Im}\, e^{in\theta}\, 2\cos\theta$$

$$\boxed{\omega = 2\omega_0 \sin\frac{\theta}{2}}$$

2002

JJ2

$f_n = \sin(n\theta)$

$f_0 = 0$

$f_N = \sin(N\theta) = 0 \Rightarrow N\theta = \pi m$

$$f_n = \sin\left(\frac{\pi m n}{N}\right)$$

$n =$ mass
$m =$ mode #

$$\omega_m = 2\omega_0 \sin\left(\frac{\pi m}{2N}\right)$$

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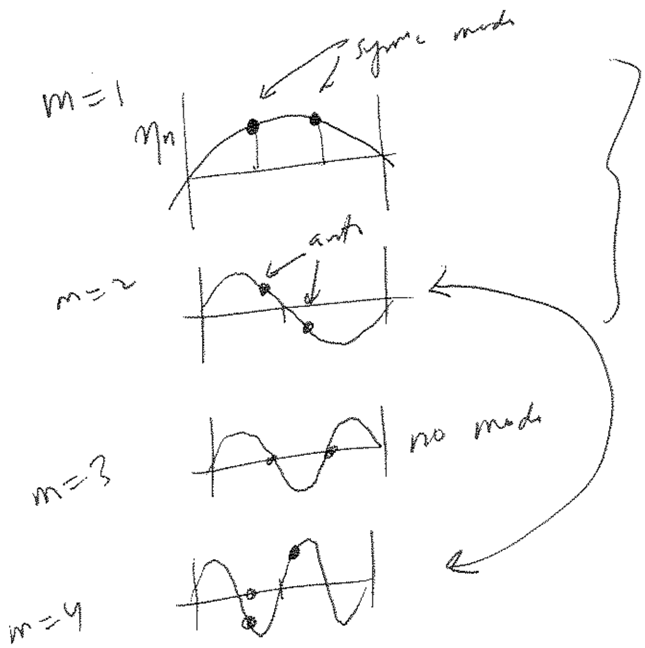
eg 2 masses $N = 3$

$\omega_1 = 2\omega_0 \sin\left(\frac{\pi}{6}\right) = \omega_0 = \sqrt{\frac{k}{m}}$

$\omega_2 = 2\omega_0 \sin\left(\frac{\pi}{3}\right) = \sqrt{3}\,\omega_0 = \sqrt{\frac{3k}{m}}$



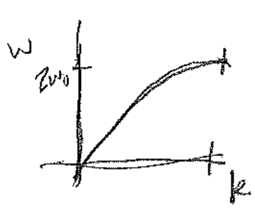
only $m=1$ & $m=2$ are physical

$$\omega_m \equiv 2\omega_0 \sin\left(\frac{\pi m}{2N}\right),$$

$m = 1, \ldots, N$

only these are physical

let $\lambda =$ "wavelength" of mode $= \frac{2L}{m}$, $k = \frac{2\pi}{\lambda} = \frac{\pi m}{L}$



$$\omega = 2\omega_0 \sin\left(\frac{kL}{2N}\right)$$

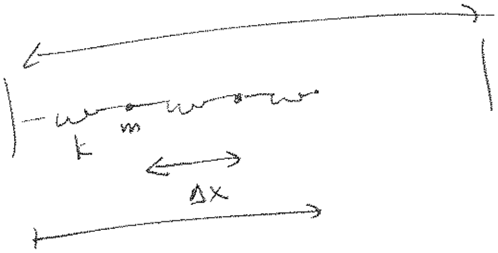
dispersion relation (between mode frequency and ~~spatial~~ spatial wave number)

Continuum limit ($N \to \infty$)

Aug 2002

JJ3

L



$\Delta x = \frac{L}{N}$

$m \sim \Delta x$

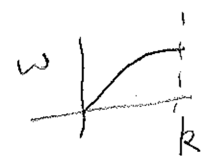
Let $\mu = \frac{\text{mass}}{\text{length}} \Rightarrow m = \mu \Delta x$

$k \sim \frac{1}{\Delta x}$ (cut in half, double k)

Let $k = \frac{K}{\Delta x} \left(= \frac{\tau}{\Delta x}\right)$

Then $\omega_0 = \sqrt{\frac{k}{m}} = \frac{1}{\Delta x}\sqrt{\frac{K}{\mu}} = \frac{N}{L}\sqrt{\frac{K}{\mu}}$

$\omega = 2\omega_0 \sin\left(\frac{kL}{2N}\right) = 2\sqrt{\frac{K}{\mu}}\left(\frac{N}{L}\right)\sin\left(\frac{kL}{2N}\right)$



$$\mu \Delta x \, \ddot{\eta}_n = +\frac{K}{\Delta x}\left(-2\eta_n + \eta_{n+1} + \eta_{n-1}\right)$$

Let $\eta_n = \eta(x)$ where $x = n\Delta x$

$$\eta_{n+1} = \eta(x+\Delta x) = \eta(x) + \eta'(x)\,\Delta x + \frac{1}{2}\eta''(x)\,\Delta x^2 + \dots$$

$$\eta_{n-1} = \eta(x-\Delta x) = \eta(x) - \eta'(x)\,\Delta x + \frac{1}{2}\eta''(x)\,\Delta x^2$$

$$-2\eta_n + \eta_{n+1} + \eta_{n-1} = \eta''(x)\,\Delta x^2$$

$$\mu\,\Delta x\,\ddot{\eta}_n = \frac{K}{\Delta x}\,\eta''(x)\,\Delta x^2$$

$$\boxed{\mu\ddot{\eta} - K\eta'' = 0}$$ ← wave eqn ...

$N \to \infty$, $\sin\left(\frac{kL}{2N}\right) \to \frac{kL}{2N}$

$\omega \to \left(\sqrt{\frac{K}{\mu}}\right)k$

dispersion relation simplifies

Infinite line of coupled oscillators (Kenyon exam)

Pdf file generated on March 23, 2017 by Steve Naculich.

Consider a row of infinitely many identical pendula, spaced a distance d apart. Each pendulum is coupled to its neighbor by springs of constant k. The Lagrangian for small oscillations is given by

$$\begin{aligned}\mathcal{L} = &\tfrac{1}{2}mL^2\left(\cdots + \dot\theta_{n-1}^2 + \dot\theta_n^2 + \dot\theta_{n+1}^2 + \cdots\right)\\ &- \tfrac{1}{2}mgL\left(\cdots + \theta_{n-1}^2 + \theta_n^2 + \theta_{n+1}^2 + \cdots\right)\\ &- \tfrac{1}{2}kL^2\left(\cdots + (\theta_{n-1}-\theta_n)^2 + (\theta_n - \theta_{n+1})^2 + \cdots\right)\end{aligned} \tag{1}$$

Euler-lagrange equations take the form

$$\ddot\theta_n + \frac{g}{L}\theta_n + \frac{k}{m}\left(2\theta_n - \theta_{n-1} - \theta_{n+1}\right) = 0 \tag{2}$$

Consider normal mode solutions $\theta_n(t) = e^{i\omega t} f_n$. This gives rise to the eigenvalue equation

$$\mathbf{M}\,\mathbf{v} \equiv \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{g}{L}+\frac{2k}{m} & -\frac{k}{m} & 0 & \cdots \\ \cdots & -\frac{k}{m} & \frac{g}{L}+\frac{2k}{m} & -\frac{k}{m} & \cdots \\ \cdots & 0 & -\frac{k}{m} & \frac{g}{L}+\frac{2k}{m} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \cdots \\ f_{n-1} \\ f_n \\ f_{n+1} \\ \cdots \end{pmatrix} = \omega^2 \begin{pmatrix} \cdots \\ f_{n-1} \\ f_n \\ f_{n+1} \\ \cdots \end{pmatrix} \tag{3}$$

Define the translation operator

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

and observe that $\mathbf{T}$ and $\mathbf{M}$ commute. This reflects the invariance of the Lagrangian under translations by a. Commuting matrices can be simultaneously diagonalized, so let's consider the eigenvalue equation $\mathbf{T}\mathbf{v} = \lambda\mathbf{v}$ which is equivalent to $f_{n+1} = \lambda f_n$, so that $f_n = \lambda^n f_0$ for complex λ. If we want the oscillations to be small then $|\lambda| = 1$ i.e. $\lambda = e^{i\phi}$. The eigenvalue equation for $\mathbf{M}$ then implies

$$\omega^2 = \frac{g}{L} + \frac{2k}{m} - \frac{k}{m}(e^{i\phi} + e^{-i\phi}) = \frac{g}{L} + \frac{2k}{m}(1 - \cos\phi) \tag{5}$$

If the chain is periodic with period N then $\phi = 2\pi m/N$, but otherwise ϕ is completely arbitrary.

Thomas got these old exams from Kenyon when they asked him to be outside examiner

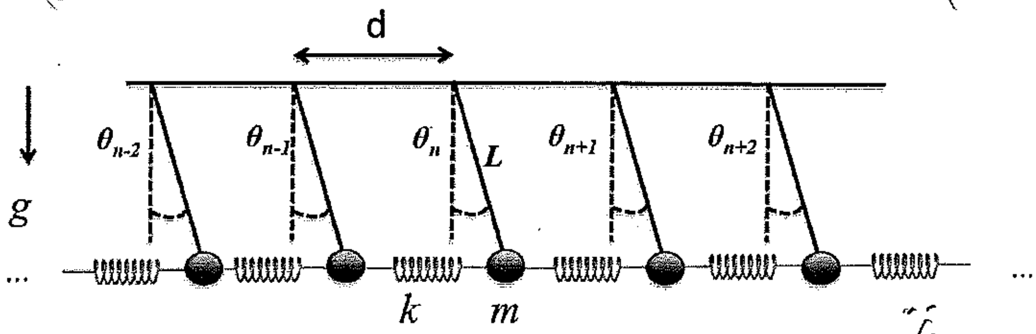
$T = e^{ikd}$

Problem 4

e^{ikd}

$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix} e^{i\omega t}$

$f_n = e^{ina} g_a$



$\ddot{\theta}_n = -\omega_0^2 \theta_n$

$+ c(\theta_{n-1} - 2\theta_n + \theta_{n+1})$

e^{ina} $\cancel{\theta_n - \theta_{n-1}}$

FIG. 1: A line of infinitely many, identical, coupled pendula

Consider a row of infinitely many identical pendula, spaced a distance d apart. Each pendulum is coupled to its neighbors by springs of constant k as shown in Fig. 1.

$\omega^2 = \omega_0^2 + c$

a) Show that, for small oscillations, the Lagrangian for this system is given by

$$\begin{aligned} L =& \frac{1}{2} mL^2 \left(\ldots \dot{\theta}_{n-2}^2 + \dot{\theta}_{n-1}^2 + \dot{\theta}_n^2 + \dot{\theta}_{n+1}^2 + \dot{\theta}_{n+2}^2 + \ldots \right) \\ & - \frac{1}{2} mgL \left(\ldots + \theta_{n-2}^2 + \theta_{n-1}^2 + \theta_n^2 + \theta_{n+1}^2 + \theta_{n+2}^2 + \ldots \right) \\ & - \frac{1}{2} kL^2 \left[\ldots + (\theta_{n-2} - \theta_{n-1})^2 + (\theta_{n-1} - \theta_n)^2 + (\theta_n - \theta_{n+1})^2 + (\theta_{n+1} - \theta_{n+2})^2 + \ldots \right] \end{aligned} \tag{4}$$

$f e^{i\omega t}$

$\theta_n(t) = e^{ina}\, \theta(t) \quad e^{i\omega}$

b) Describe the symmetry of the above system, and use it to argue that the eigenvectors of the above system must be of the form

$e^{\frac{2\pi i}{a}}$

$$\vec{u} = [\ldots, 1, \lambda, \lambda^2, \lambda^3, \ldots]^T, \tag{5}$$

$\begin{pmatrix} \ddot{\theta}_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} =$

$m l^2 \ddot{\theta}_n$

where $\lambda = e^{i\kappa d}$, $\kappa = 2\pi/(ad)$ $a \in \{1, 2, 3, \ldots\}$. This is incorrect

$= -mgl\theta_n$

$-kL^2 (\theta_n - \theta_{n+1}) + (\theta_n - \theta_{n-1})$

c) Show that the allowed frequencies of this system are given by

$$\omega = \sqrt{\frac{g}{l} + \frac{2k}{m}(1 - \cos(\kappa d))} \tag{6}$$

$\omega^2 = \omega_0^2 + c\left(e^{ia} - 2 + e^{-ia}\right)$

d) Sketch the normal modes for $a = 1$ and $a = 2$.