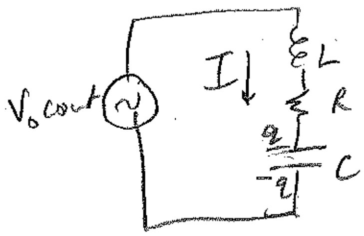
mechanical damped driven oscillator $m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t$

[They've seen RLC circuits in the problems]

RLC circuit w/ AC driving voltage



Let q = charge on capacitor (upper plate)

$$I = \frac{dq}{dt}$$

$$V_L + V_R + V_C = V_0 \cos\omega t$$

$$L\frac{dI}{dt} + RI + \frac{q}{C} = V_0 \cos\omega t$$

$$L\ddot{q} + R\dot{q} + \left(\frac{1}{C}\right)q = V_0 \cos\omega t$$

Analogy

mech	electrical
x	q
v	I
m	L
b	R
k	$1/C$
$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
$\gamma = \frac{b}{m}$	$\gamma = \frac{R}{L}$

$$Q = \frac{\omega_0}{\gamma}$$

complexify: $L\ddot{q} + R\dot{q} + \frac{1}{C}q = V_0 e^{i\omega t}$

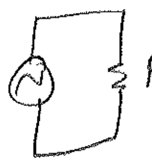
Impedance Z ≡ the response of the current to an AC voltage

$$I = \frac{V_0 e^{i\omega t}}{Z}$$ [then take real part]

← this eqn defines Z

$|Z|$ describes the "resistance" and the phase of Z describes how much the current leads or lags the driving voltage

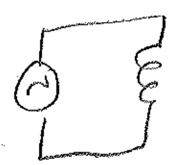
① resistor



$V_R = IR = V_0 e^{i\omega t}$

$I = \frac{V_0 e^{i\omega t}}{R} \Rightarrow Z_R = R$

② inductor



$V_L = L\frac{dI}{dt} = V_0 e^{i\omega t}$

$\frac{dI}{dt} = \frac{V_0}{L} e^{i\omega t}$

$I = \frac{V_0}{i\omega L} e^{i\omega t} \Rightarrow Z_L = i\omega L$

$= \frac{V_0}{\omega L} e^{i(\omega t - \frac{\pi}{2})}$

↑ current lags voltage by 90°

③ capacitor

$V_C = q/C = V_0 e^{i\omega t}$

$q = C V_0 e^{i\omega t}$

$I = i\omega C V_0 e^{i\omega t} = \frac{V_0 e^{i\omega t}}{\left(\frac{1}{i\omega C}\right)} \Rightarrow Z_C = \frac{1}{i\omega C}$

$= \omega C V_0 e^{i(\omega t + \frac{\pi}{2})}$

↑ current leads voltage by 90°

Impedances of elements in series add $Z = Z_1 + Z_2$

" " " " parallel add inversely $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$

RLC circuit: $Z = Z_R + Z_L + Z_C = R + i\omega L - \frac{i}{\omega C}$

mechanical impedance Z characterizes the response of the (steady state) <u>velocity</u> to a harmonic driving force

$$\dot{x} = \frac{F_0 e^{i\omega t}}{Z}$$

Recall $m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}$

<u>steady state</u> solution $x = Ce^{i\omega t}$

$$\Rightarrow C = \frac{F_0}{-m\omega^2 + ib\omega + k}$$

$$\dot{x} = i\omega C e^{i\omega t}$$

$$= \frac{F_0 e^{i\omega t}}{\left(im\omega + b - i\frac{k}{\omega}\right)}$$

$$\Rightarrow Z = b + im\omega - \frac{ik}{\omega}$$

$\xrightarrow[\text{analog}]{\text{electrical}}$ $Z = \underbrace{R + i\omega L + \frac{1}{i\omega C}}_{R,L,C \text{ in series}}$

$$= b\left[1 + i\left(\frac{m}{b}\omega - \frac{k}{b\omega}\right)\right]$$

$$= b\left[1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

Recall $\begin{cases} \omega_0 = \sqrt{\frac{k}{m}} \\ Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{km}}{b} \end{cases}$

$$|Z| = b\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$$, minimized at $\omega = \omega_0$

$$\dot{x} = \mathrm{Re}\left[\frac{F_0}{Z} e^{i\omega t}\right]$$

$$= \mathrm{Re}\left[\frac{F_0}{|Z|} e^{i(\omega t + \phi + \frac{\pi}{2})}\right]$$

velocity leads displacement by $\frac{\pi}{2}$ since $\dot{x} = i\omega C e^{i\omega t}$

$$= \frac{F_0}{|Z|}\cos\left(\omega t + \phi + \frac{\pi}{2}\right)$$

Amplitude of velocity $= \dfrac{F_0}{|Z|}$, maximized at $\omega = \omega_0$

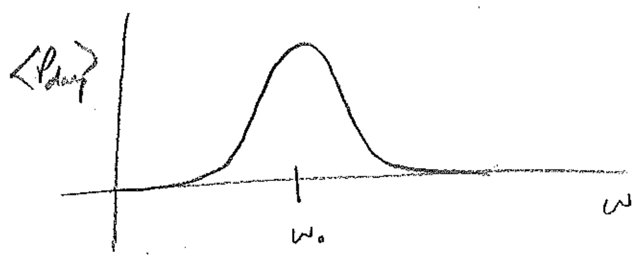
<u>Power dissipated by a damped driven oscillator (steady state)</u>

work done by damping force $dW = \vec{F}_{damp} \cdot d\vec{r}$

power dissipated $P = \dfrac{dW}{dt} = \vec{F}_{damp} \cdot \dfrac{d\vec{r}}{dt} = (-b\dot{x})\,\dot{x} = -b\,\dot{x}^2$

$$P = -b\frac{F_0^2}{|Z|^2}\cos^2\left(\omega t + \phi + \frac{\pi}{2}\right)$$

$$\langle P_{damp}\rangle = -\frac{bF_0^2}{2|Z|^2}$$

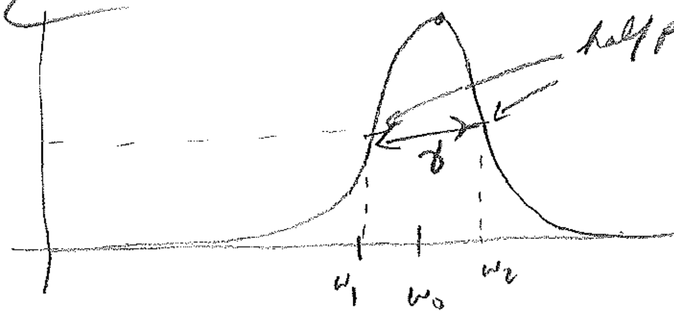


max power dissipated at $\omega = \omega_0$ when $|Z| = b \Rightarrow \langle P_{damp}\rangle = -\dfrac{F_0^2}{2b}$

[seems surprising that this is <u>large</u> for <u>lightly</u> damped
→ due to large velocity at resonance]

NOT FOR CLASS

J



half power points

γ = width (FWHM)

similar to problem 6 (probset 5)

$$\langle P \rangle = \frac{T_0^2}{2b\left[1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2\right]}$$

At half power points [ ] = 2

$$Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = 1$$

$$\frac{\omega^2 - \omega_0^2}{\omega\omega_0} = \pm\frac{1}{Q}$$

$$\omega^2 \mp \frac{\omega_0}{Q}\omega - \omega_0^2 = 0$$

$$\frac{(\omega - \omega_0)(\omega + \omega_0)}{\omega\omega_0} = \pm\frac{1}{Q}$$

$$\omega = \frac{1}{2}\left[\pm\frac{\omega_0}{Q} + \sqrt{\frac{\omega_0^2}{Q^2} + 4\omega_0^2}\right]$$

$\omega_2 > \omega_0$ so

$$\frac{(\omega_2 - \omega_0)(\omega_2 + \omega_0)}{\omega_2\omega_0} = \frac{1}{Q}$$

$$\Delta\omega = \frac{\omega_0}{Q} = \underline{\underline{\gamma}}$$

$\omega_1 < \omega_0$ so

$$\frac{(\omega_1 - \omega_0)(\omega_1 + \omega_0)}{\omega_1\omega_0} = -\frac{1}{Q}$$