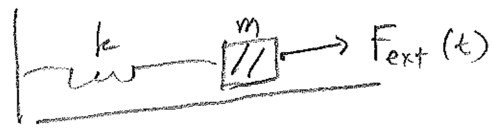
Driven oscillator



$$m\ddot{x} = -kx - b\dot{x} + F_{ext}(t)$$

restoring damping specified arbitrary external force

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{1}{m}F_{ext}(t) \qquad \text{inhomogeneous}$$

general solution $x = x_c + x_p$

complementary particular

if underdamped $\Rightarrow$ $x_c = A_f\, e^{-\frac{\gamma t}{2}} \cos(\omega_f t + \phi_f)$

undetermined coeffs

$x_p(t)$ depends on form of $F_{ext}(t)$

Example: harmonic external force w/ frequency ω

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

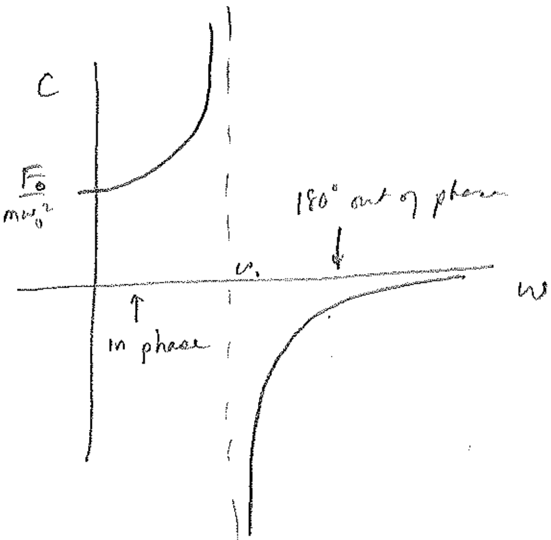
First consider <u>undamped</u> driven oscillator

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

Try $x_p = C \cos \omega t$

$$(-\omega^2 + \omega_0^2)\, C \cos \omega t = \frac{F_0}{m} \cos \omega t$$

$$C = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$



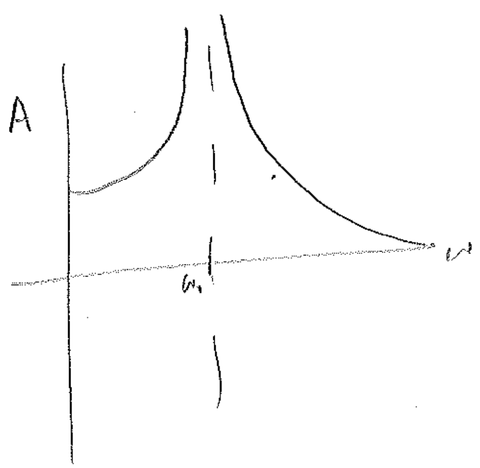
$\omega < \omega_0 \Rightarrow C$ has same sign as F_0 (in phase)

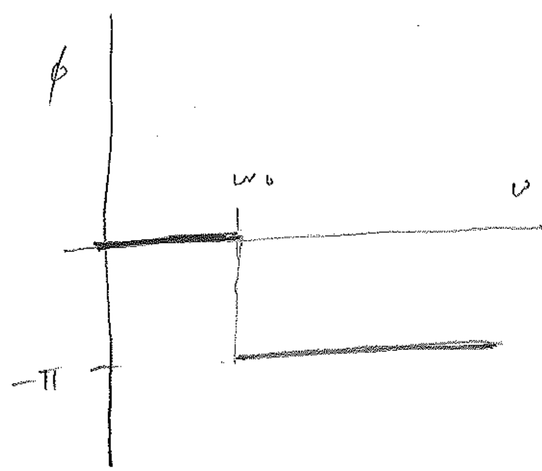
$\omega > \omega_0 \Rightarrow C$ has opposite sign

Let $A = |C| = \frac{F_0}{m|\omega_0^2 - \omega^2|}$

and $x_p = A \cos(\omega t + \phi)$ where $\phi = \begin{cases} 0 & \omega < \omega_0 \\ -\pi & \omega > \omega_0 \end{cases}$

since $\cos(\omega t - \pi) = -\cos \omega t$





Damping

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Try $x_p = A\cos\omega t$

Doesn't work because of $\gamma\dot{x}$ term

Try $x_p = A\cos\omega t + B\sin\omega t$

This will work, but let's try an easier approach

Consider a complexified source

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$$

Try $x_p = Ce^{i\omega t}$ ✓

$$(-\omega^2 + i\gamma\omega + \omega_0^2)\, C\cancel{e^{i\omega t}} = \frac{F_0}{m}\cancel{e^{i\omega t}}$$

$$C = \frac{F_0/m}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

Write this in polar form:

$$C = Ae^{i\phi}$$

$$\begin{cases} A = \dfrac{\frac{F_0}{m}}{\sqrt{(\omega_0^2-\omega^2) + (\gamma\omega)^2}} \\ \\ \phi = -\arctan\left(\dfrac{\gamma\omega}{\omega_0^2-\omega^2}\right) \end{cases}$$

NB. in particular solution A + ϕ are <u>not</u> arbitrary constants; they are determined by the driving force

We have found that

$$x_p = A e^{i(\omega t + \phi)}$$ is a particular solution of

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$$

Therefore

$$x_p = \mathrm{Re}\left[A e^{i(\omega t + \phi)}\right]$$ is a particular solution of

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \mathrm{Re}\left[e^{i\omega t}\right] = \frac{F_0}{m} \cos\omega t$$

(decomposition principle for real linear inhomogen eqns)

$$x_p = A \cos(\omega t + \phi)$$

if y is a complex solution of a real linear homogeneous o.d.e. of complex source $s(x)$

$$Ly = s(x)$$

then $\mathrm{Re}\, y$ is a soln of source $\mathrm{Re}\, s(x)$
& $\mathrm{Im}\, y$ " " " " $\mathrm{Im}\, s(x)$

[Proof: $y = \mathrm{Re}\, y + i \mathrm{Im}\, y$
$s = \mathrm{Re}\, s + i \mathrm{Im}\, s$
separate into real + imag parts]

General solut.

$$X = X_p + X_c$$

$$= \underbrace{A \cos(\omega t + \phi)}_{\substack{\text{steady-state solution} \\ \text{(periodic)}}} + \underbrace{A_f e^{-\frac{\gamma t}{2}} \cos(\omega_f t + \phi_f)}_{\substack{\text{transient solution} \\ (\to 0 \text{ as } t \to \infty)}}$$

$$A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}, \quad \phi = -\arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right)$$

A_f, ϕ_f are arbitrary constants, determined by initial conditions

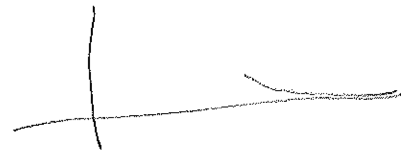
Note: steady state solution has same frequency as the driving force (not ω_f)

Note: no arbitrary constants in the steady state solution so $t \to \infty$ behavior independent of initial conditions

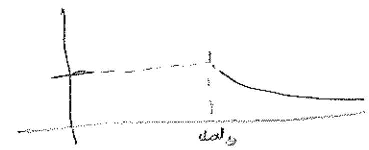
[DEMO]

Consider amplitude A of steady-state solution

$\omega \to \infty$: $A \xrightarrow[\omega\to\infty]{} \dfrac{F_0}{m\omega^2} \to 0$

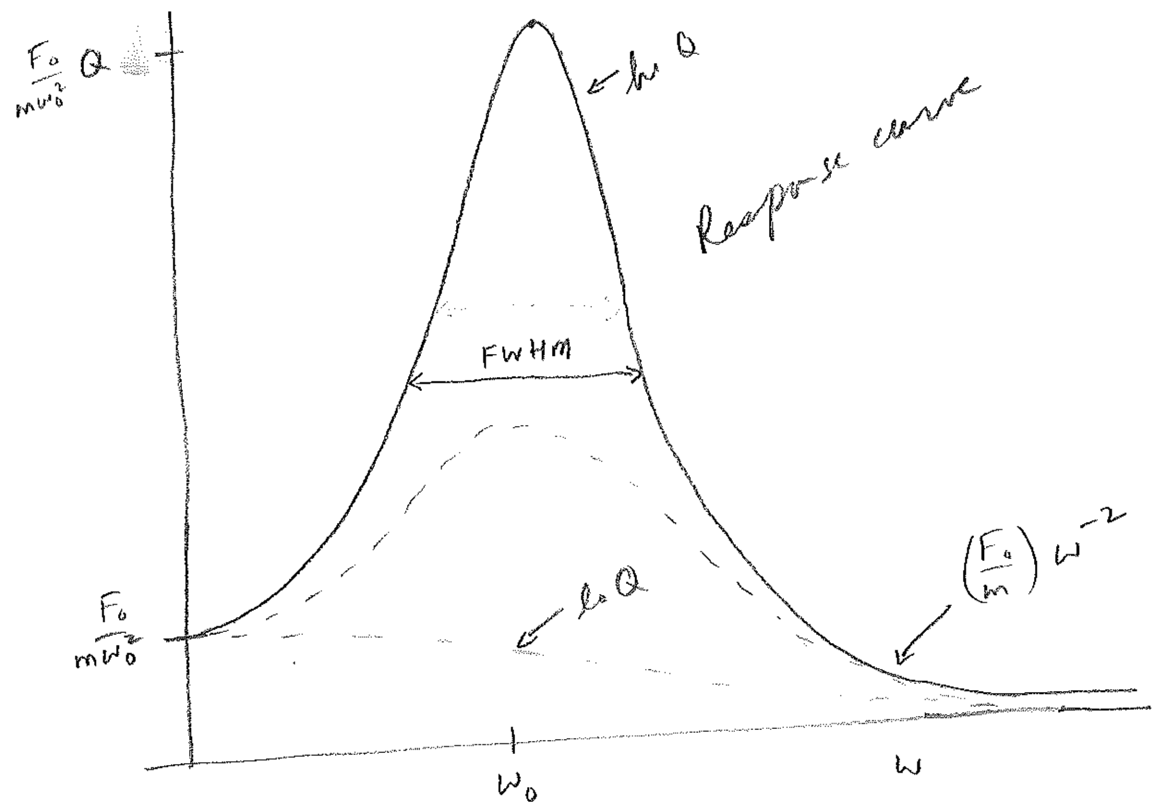


$\omega \to 0$: $A \xrightarrow[\omega\to 0]{} \dfrac{F_0}{m\omega_0^2} = \text{const}$



[what happens at $\omega = \omega_0$?]

$\omega \to \omega_0$: $A \xrightarrow[\omega\to\omega_0]{} \dfrac{F_0}{m\gamma\omega} = \dfrac{F_0}{m\gamma\omega_0} = \dfrac{F_0}{m\omega_0^2}\dfrac{\omega_0}{\gamma} = \dfrac{F_0}{m\omega_0^2} Q$

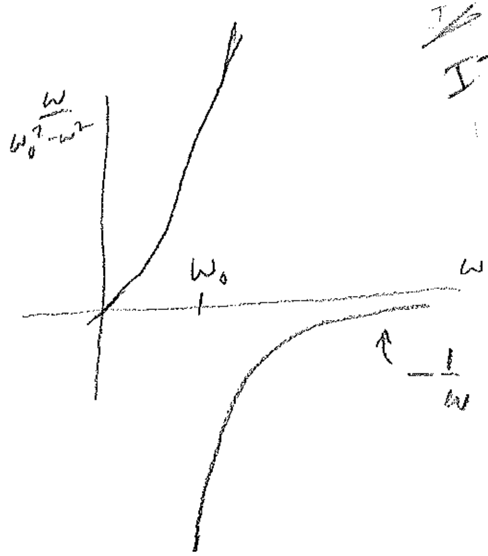


For ~~small~~ large Q,
large response to force when $\omega \approx \omega_0 \Rightarrow$ resonance

As $Q \uparrow$, peak $\uparrow$ and width $\downarrow$

Consider phase

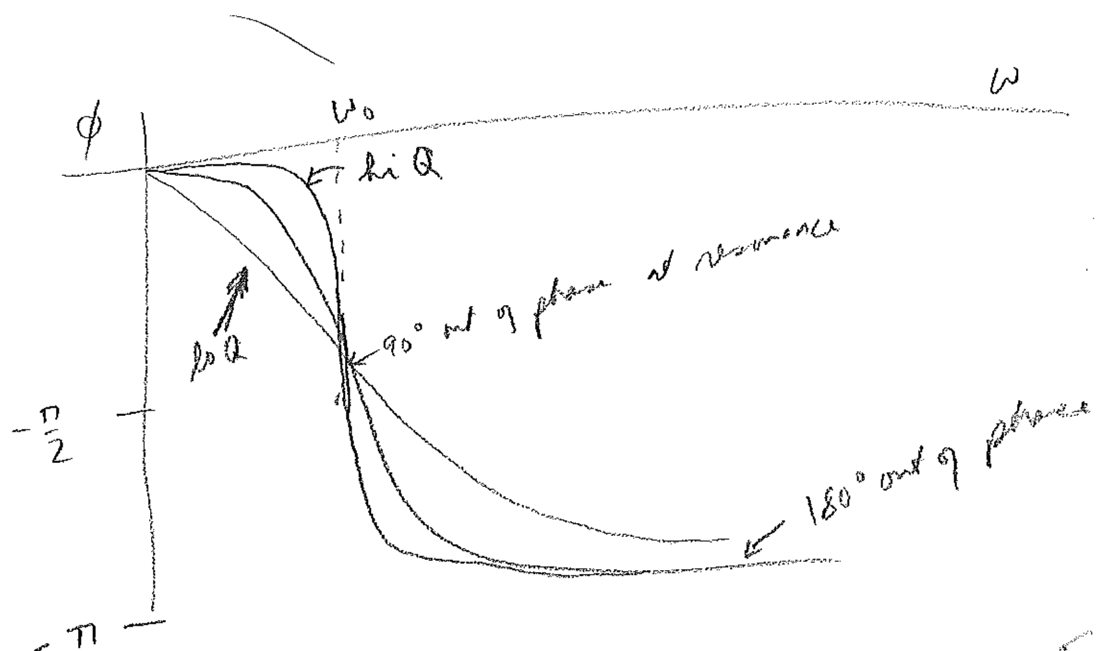
$$\phi = -\arctan\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right)$$



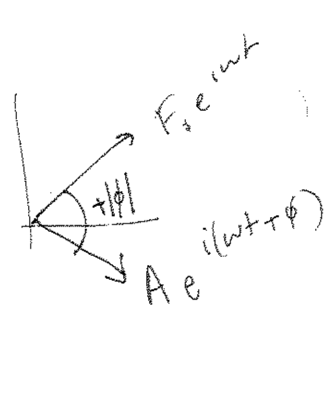
$$\phi \xrightarrow[\omega\to 0]{} 0$$

$$\phi \xrightarrow[\omega\to\omega_0]{} -\arctan(\infty) = -\frac{\pi}{2}$$

$$\phi \xrightarrow[\omega\to\infty]{} -\arctan\left(-\frac{1}{\omega}\right) = -\pi$$



phase of x lags phase $F_0 e^{i\omega t}$



[I8]

DEMO

- Watch black and white arrows to see phase difference

[when change frequency turn up damping to kill off transient, then turn down to see hi Q behavior]

$$f \approx \left[\frac{\text{meter reading}}{39 \text{ or } 40}\right] \text{Hz}$$

$f_0 \approx \frac{1}{2}$ Hz (about 19–20 on meter)

[skip in 2017]

I [skip in 2017]

~~optional~~

Consider various limits physically

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}$$

$$\cancel{\ddot{x} + \beta\dot{x} + \omega^2 x = \frac{F_0}{m} e^{i\omega t}}$$

Ansatz $x = Ce^{i\omega t}$

$\dot{x} = i\omega C e^{i\omega t}$

$\ddot{x} = -\omega^2 C e^{i\omega t}$

$\Rightarrow C = \dfrac{F_0}{k + ib\omega - m\omega^2} = Ae^{i\phi}$

$Re(F_0 e^{i\omega t})$

$\Rightarrow Re(Ce^{i\omega t})$

$x = A\cos(\omega t + \phi)$

(1) $\omega \to 0 \Rightarrow$ velocity + accel. are small $\Rightarrow$ damping + inertia irrelevant

"stiffness controlled"

$kx = F_0 e^{i\omega t}$

$x = \dfrac{F_0}{k} e^{i\omega t} \Rightarrow A = \dfrac{F_0}{k}$ ✓

x + in phase w/ $F_0 \Rightarrow \phi = 0$

(2) $\omega \to \infty \Rightarrow$ acceleration huge $\Rightarrow$ "inertia controlled"

"vibration isolation"

$m\ddot{x} = F_0 e^{i\omega t}$

$x = \dfrac{F_0}{-\omega^2 m} e^{i\omega t} = \dfrac{F_0}{m\omega^2} e^{i(\omega t - \pi)}$

so $A = \dfrac{F}{m\omega^2}$, $\phi = -\pi$

x lags force by 180°

(3) $\omega \to \omega_0$ $\quad \underbrace{m\ddot{x}}_{-m\omega^2 e^{i\omega t}} + \underbrace{kx}_{ke^{i\omega t}}$ cancel $\Rightarrow$ "resistance limited"

$b\dot{x} = F_0 e^{i\omega t}$

$x = \dfrac{F_0}{i\omega b} e^{i\omega t} = \left(\dfrac{F_0}{\omega b}\right) e^{i(\omega t - \frac{\pi}{2})}$

A lags by 90°

Amplification (resonance)

$b \neq 0$ keeps A from blowing up!

I ~~III~~ ~~IV~~

(~~possibly~~ skip)

If apply force ~~by~~ $F_0 e^{i\omega t}$ by moving anchor pt by $\frac{F_0}{k} e^{i\omega t}$:

Consider ratio ~~mass~~ displacement amplitude to anchor pt: $\dfrac{A}{\left(\frac{F_0}{k}\right)}$

$\omega \to 0$: $\dfrac{A}{\left(\frac{F_0}{k}\right)} = \dfrac{F_0/m\omega_0^2}{F_0/k} \to 1$

mass moves w/ anchor pt

$\omega \to \omega_0$ $\dfrac{A}{\left(\frac{F_0}{k}\right)} = \dfrac{\left(\frac{F_0}{\omega_0 b}\right)}{\left(\frac{F_0}{k}\right)} = Q$

anchor pt motion is amplified (resonance)

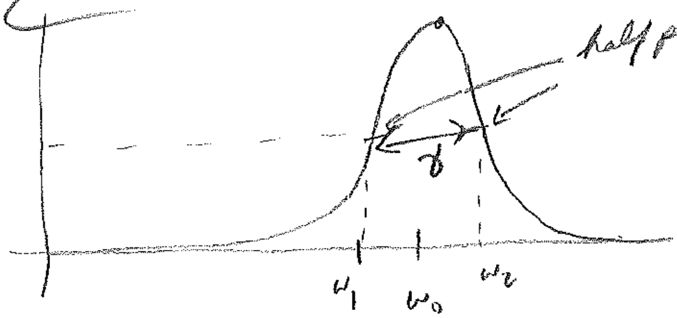
$\omega \to \infty$ $\dfrac{A}{\left(\frac{F}{k}\right)} = \dfrac{\frac{F}{m\omega^2}}{\frac{F}{m\omega_0^2}} = +\frac{\omega_0^2}{\omega^2} \longrightarrow 0$

vibration isolation

NOT FOR CLASS

I

half power points



γ = width (FWHM)

Similar to problem 6 (problem set 5)

$$\langle P \rangle = \frac{T_0^{\,2}}{2b\left[1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2\right]}$$

At half power points $[\;\;] = 2$

$$Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = 1$$

$$\frac{\omega^2 - \omega_0^2}{\omega\omega_0} = \pm\frac{1}{Q}$$

$$\omega^2 \mp \frac{\omega_0}{Q}\omega - \omega_0^2 = 0$$

$$\frac{(\omega-\omega_0)(\omega+\omega_0)}{\omega\omega_0} = \pm\frac{1}{Q}$$

$$\omega = \frac{1}{2}\left[\pm\frac{\omega_0}{Q} + \sqrt{\frac{\omega_0^2}{Q^2} + 4\omega_0^2}\right]$$

$\omega_2 > \omega_0$ so

$$\frac{(\omega_2-\omega_0)(\omega_2+\omega_0)}{\omega_2\omega_0} = \frac{1}{Q}$$

$$\Delta\omega = \frac{\omega_0}{Q} = \gamma$$

$\omega_1 < \omega_0$ so

$$\frac{(\omega_1-\omega_0)(\omega_1+\omega_0)}{\omega_1\omega_0} = -\frac{1}{Q}$$