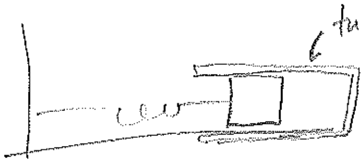
Oscillator w/ viscous damping



Assume a velocity-dependent damping force $F_{damp}(v)$

$F_{damp}(0) = 0$

Taylor series: $F_{damp}(v) = -b_1 v - b_2 v^2 + \dots$

small oscillations $\Rightarrow$ small v $\Rightarrow$ $F_{damp} \approx -bv = -b\dot{x}$

$$m\ddot{x} = F_{net} = -kx^2 - b\dot{x}$$

$$\boxed{\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0}$$

Recall $\frac{k}{m} = \omega_0^2$ where $\omega_0 =$ natural frequency of undamped

Define $\frac{b}{m} = \gamma$, damping coefficient.

Observe ω_0, γ both have units s^{-1}!

Define dimensionless ratio $\frac{\omega_0}{\gamma} = Q$ (quality factor)

Light damping $\Rightarrow$ $\gamma \ll \omega_0$ $\Rightarrow$ $Q \gg 1$ (high quality)

(undamped $\Rightarrow$ $Q = \infty$)

$\ddot{x} + \gamma \dot{x} + \omega_0^2 x \Rightarrow$ 2nd order linear o.d.e.

exponential ansatz $x = e^{pt}$

$p^2 \cancel{e^{pt}} + \gamma p \cancel{e^{pt}} + \omega_0^2 \cancel{e^{pt}} = 0$

$p^2 + \gamma p + \omega_0^2 = 0 \Leftarrow$ 2nd order algebraic equation

$p_\pm = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$ 2 roots

$x = e^{p_+ t}$ and $e^{p_- t}$ 2 solutions (unless roots are degenerate)

Discriminant $\Delta \equiv \frac{\gamma^2}{4} - \omega_0^2 = \left(\frac{\gamma}{2}\right)^2 [1 - 4Q^2]$

$4Q^2 = 1 - \frac{4\Delta}{\gamma^2}$

$Q = \frac{1}{2}\sqrt{1 - \frac{4\Delta}{\gamma^2}}$

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3 cases:

① $\Delta < 0 \Rightarrow Q > \frac{1}{2}$ "under damped"

($Q = \infty$, undamped)

② $\Delta > 0 \Rightarrow Q < \frac{1}{2}$ "overdamped"

③ $\Delta = 0 \Rightarrow Q = \frac{1}{2}$ "critically damped"

① underdamped: $(\Delta < 0,\ Q > \frac{1}{2})$

$$p = -\frac{\gamma}{2} \pm \sqrt{-\underbrace{\left(\omega_0^2 - \frac{\gamma^2}{4}\right)}_{\omega_f^2}}$$

$$= -\frac{\gamma}{2} \pm i\omega_f$$

$$\omega_f = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}$$

"free

free natural

$\omega_f \le \omega_0$

$$x = \frac{1}{2} c e^{-\frac{\gamma}{2}t} e^{i\omega_f t} + \frac{1}{2} d e^{-\frac{\gamma}{2}t} e^{-i\omega_f t}$$

complex

Require $x = x^*$ $\Rightarrow$ $d = c^*$

$$x = e^{-\frac{\gamma}{2}t} \underbrace{\left[\frac{1}{2} c e^{i\omega_f t} + \frac{1}{2} c^* e^{-i\omega_f t}\right]}_{\mathrm{Re}[c e^{i\omega_f t}]}$$

form III

Short cut: write only one of the complex solutions

$$x = c e^{-\frac{\gamma}{2}t} e^{i\omega_f t}$$

and use principle of decomposition to take the real part

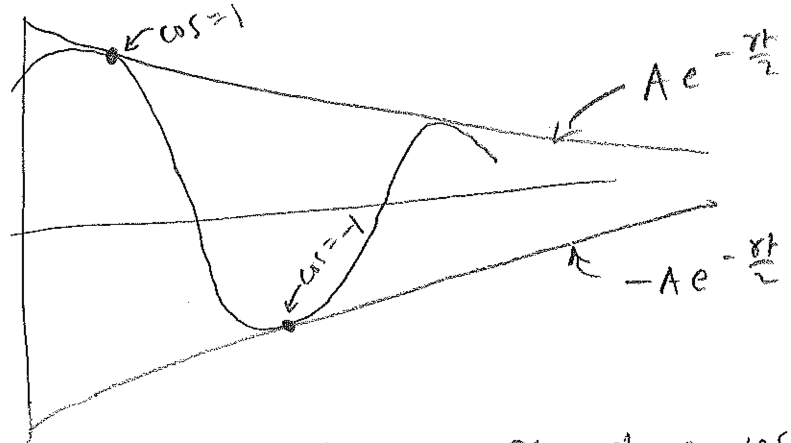
$$x = \mathrm{Re}\left[c e^{-\frac{\gamma}{2}t} e^{i\omega_f t} \right] \quad \leftarrow \text{same as before}$$

c ← complex # so 2 arbitrary constants $c = Ae^{i\phi}$

$$x = \mathrm{Re}\left[A e^{-\frac{\gamma t}{2}} e^{i(\omega_f t + \phi)} \right] \quad \leftarrow e^{-\frac{\gamma}{2}t}: \text{ slowly changing}$$

$$= A e^{-\frac{\gamma}{2}t} \cos(\omega_f t + \phi) \quad \leftarrow \cos(\omega_f t + \phi) \text{ rapidly changing}$$

To sketch, use $\pm A e^{-\frac{\gamma}{2}t}$ as the envelope

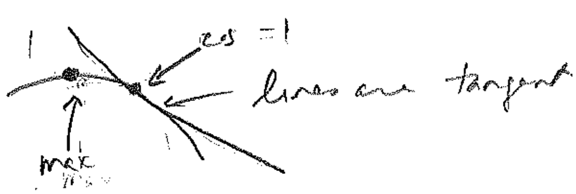


Since $|\cos(\omega_f t + \phi)| \le 1$, x stays inside the envelope

$$-Ae^{-\frac{\gamma}{2}t} \le x \le Ae^{-\frac{\gamma t}{2}}$$

x touches the envelope when $\cos(\omega_f t + \phi) = \pm 1$

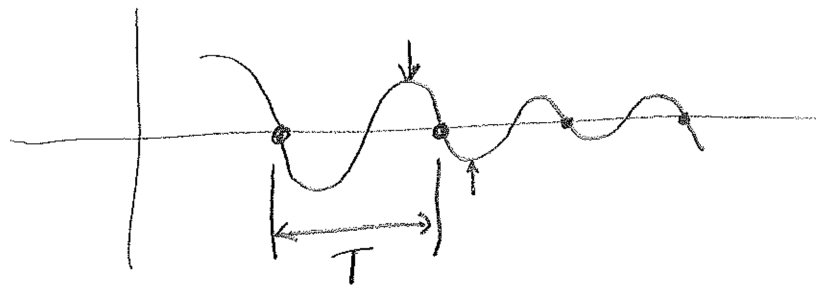
These are <u>not</u> maxima & minima (because slope $\neq 0$)



period of motion?

Not strictly periodic.

Define period as time between every other node



node $\Rightarrow$ $\cos(\omega_f t + \phi) = 0$ $\Rightarrow$ $\omega_f t + \phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

2π difference

$$(\omega_f(t+T) + \phi) - (\omega_f t + \phi) = 2\pi$$

$$\omega_f T = 2\pi$$

$$\boxed{T = \frac{2\pi}{\omega_f}}$$

Hw: Show that time between successive turning points is $\frac{T}{2}$

Hw: show that ratio of displacements at successive turning points is: $-e^{-\frac{\gamma T}{4}}$

$\Rightarrow \therefore$ ratio of successive maxima $r = e^{-\frac{\gamma T}{2}}$

$$\gamma = -\frac{2}{T} \ln r$$

called logarithmic decrement

Two time scales: period T, decay time $\tau = \frac{2}{\gamma}$

Ratio: $\frac{\tau}{T} = \frac{2}{\gamma}\left(\frac{\omega_f}{2\pi}\right) \approx \frac{\omega_0}{\pi\gamma} = \frac{Q}{\pi}$ = # periods for amplitude to decrease to e^{-1}

large Q

Energy of a damped oscillator

Recall $E_{mech} = K + U$

where $K = \frac{1}{2} m\dot{x}^2$, $U = \frac{1}{2}kx^2$ for harmonic oscillator

Conservative forces $\Rightarrow$ E_{mech} conserved

Damping forces not conservative (can't be written as $-\nabla U$)

$$m\ddot{x} + b\dot{x} + kx = 0$$

Trick: multiply by $\dot{x}$

$$m\dot{x}\ddot{x} + kx\dot{x} = -b\dot{x}^2$$

$$\frac{d}{dt}\underbrace{\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2\right)}_{E_{mech}} = -b\dot{x}^2 < 0$$

Damping causes E_{mech} to decrease

Rate of energy loss

$$\frac{dE_{mech}}{dt} = (-b\dot{x})\cdot\dot{x} = \underbrace{F_{damp}\cdot\frac{dx}{dt}}_{\text{power dissipated by damping force}}$$

Recall work done by damping

$$dW = \vec{F}_{damp}\cdot d\vec{r}$$

$$P = \frac{dW}{dt} = \vec{F}_{damp}\cdot\frac{d\vec{r}}{dt}$$

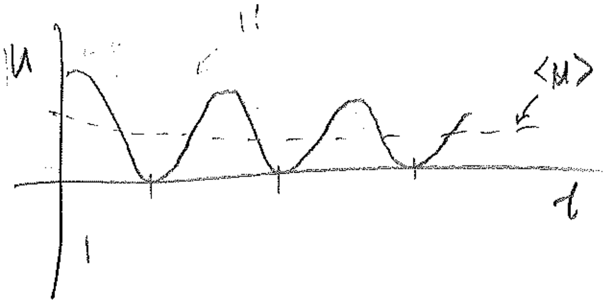
Energy of an <u>underdamped</u> oscillator

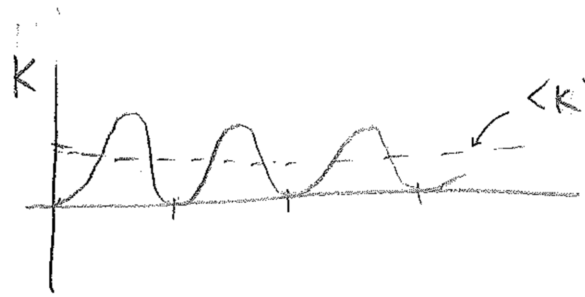
$$x = A e^{-\frac{\gamma t}{2}} \cos(\omega_f t + \phi)$$

$$\dot{x} = A e^{-\frac{\gamma t}{2}} \left[-\frac{\gamma}{2} \cos(\omega_f t + \phi) - \omega_f \sin(\omega_f t + \phi) \right]$$

$$U = \frac{1}{2} k x^2 = \frac{1}{2} m \omega_0^2 A^2 e^{-\gamma t} \cos^2(\omega_f t + \phi)$$

$$K = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m A^2 e^{-\gamma t} \left[\frac{\gamma^2}{4} \cos^2() - \gamma \omega_f \cos() \sin() + \omega_f^2 \sin^2() \right]$$





Let's smooth these out by averaging over one period

$$\langle U \rangle = \frac{1}{2} m \omega_0^2 A^2 \underbrace{\frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} e^{-\gamma t} \cos^2(\omega_f t + \phi) \, dt}_{\text{a "window" of width } T \text{ around } t}$$

assume light damping so $e^{-\gamma t}$ slowly changing relative to $\cos(\omega_f t)$

$$\approx \frac{1}{2} m \omega_0^2 A^2 e^{-\gamma t} \underbrace{\frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \cos^2(\omega_f t + \phi) \, dt}_{\frac{1}{2}} = \frac{1}{4} m \omega_0^2 A^2 e^{-\gamma t}$$

$$\langle K \rangle = \frac{1}{2} m A^2 e^{-\gamma t} \left[\frac{\gamma^2}{4} \underbrace{\langle \cos^2 \rangle}_{\frac{1}{2}} - \gamma \omega_f \underbrace{\langle \cos \cdot \sin \rangle}_{0} + \omega_f^2 \underbrace{\langle \sin^2 \rangle}_{\frac{1}{2}} \right]$$

$$= \frac{1}{4} m A^2 e^{-\gamma t} \left[\underbrace{\frac{\gamma^2}{4} + \omega_f^2}_{\omega_0^2} \right] = \frac{1}{4} m \omega_0^2 A^2 e^{-\gamma t}$$

Note $\langle U \rangle = \langle K \rangle$

$$\langle E \rangle = \langle U \rangle + \langle K \rangle = \frac{1}{2} m \omega_0^2 A^2 e^{-\gamma t}$$

Recall $\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$

$x = e^{pt}$

$\Rightarrow p = -\frac{\gamma}{2} \pm \sqrt{\underbrace{\frac{\gamma^2}{4} - \omega_0^2}_{\Delta}}$ [Demo: 0]

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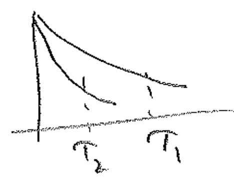
(2) overdamped : $\Delta > 0$, ie $\frac{\gamma}{2} > \omega_0$

$\Rightarrow$ 2 real roots.

General solution is linear comb. of the 2 indep solns

$$x(t) = c\, e^{-\underbrace{(\frac{\gamma}{2} - \sqrt{\Delta})}_{\frac{1}{\tau_1}} t} + d\, e^{-\underbrace{(\frac{\gamma}{2} + \sqrt{\Delta})}_{\frac{1}{\tau_2}} t}$$

motion purely damped by 2 time constants τ_1 + τ_2



Since $\tau_2 < \tau_1$, first solution dominates for large t

$$x(t) \xrightarrow[\text{large } t]{} c\, e^{-\frac{t}{\tau_1}}$$ [Demo: 15-17 volts]

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(3) critically damped : $\Delta = 0$, ie $\frac{\gamma}{2} = \omega_0$

one root:

$x(t) = c\, e^{-\frac{\gamma t}{2}}$ gives faster damping than overdamped. [Demo 10 volts]

General solution requires 2 arb. constants $\Rightarrow$ need another solution [HW]

general solution $x(t) = c\, e^{-\frac{\gamma t}{2}} + d\,($ other solution $)$