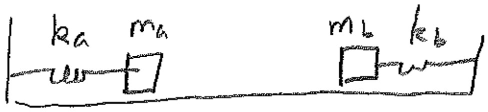
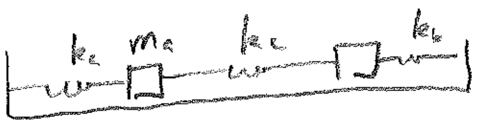
Normal modes

[Demo: chaos pendulum
① fixed, small oscillations $\ddot{\theta} \sim -\sin\theta \sim -\theta$, constant period
② larger oscillations, longer period
③ unfixed, small oscillations → 2 normal modes
④ large oscillations, chaos]



uncoupled oscillators, $\omega_a = \sqrt{\frac{k_a}{m_a}}$, $\omega_b = \sqrt{\frac{k_b}{m_b}}$



coupled oscillators: [complicated]

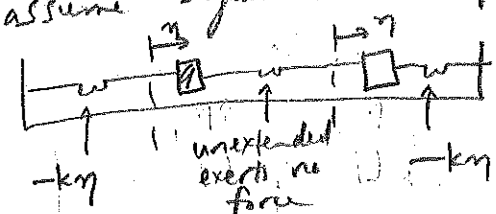
∃ normal modes for which motion is "simple"
i.e. all parts oscillate w/ same frequency $\omega^{(1)}$ or $\omega^{(2)}$

normal modes = # degrees of freedom = # 2nd order o.d.e.'s.

Simple case: $m_a = m_b = m$
$k_a = k_b = k \neq k_c$

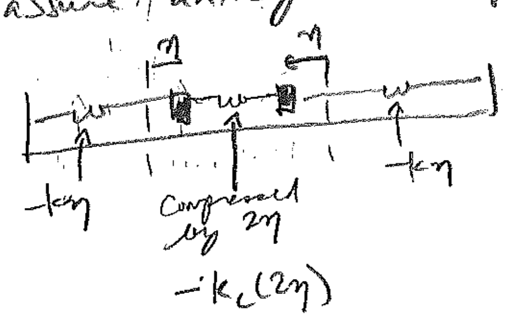
Guess the normal modes

① assume symmetric displacements



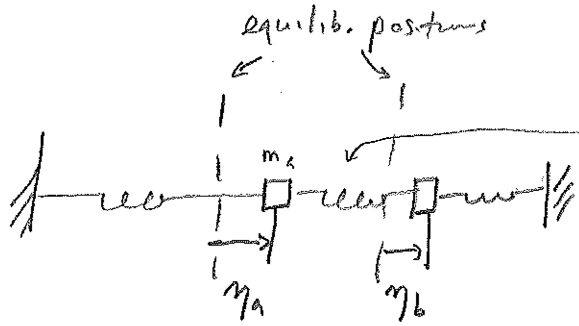
$\omega^{(1)} = \sqrt{\frac{k}{m}}$

② assume antisymmetric displacements



$F_{net} = -(k + 2k_c)\eta$

$\Rightarrow \omega^{(2)} = \sqrt{\frac{k + 2k_c}{m}}$



η_a = displacement of m_a from eqbm
η_b = displacement of m_b from eqbm

$$F_a = -k_a \eta_a - k_c(\eta_a - \eta_b)$$

$$F_b = -k_b \eta_b + k_c(\eta_a - \eta_b)$$

forces are linear in each displacement

$$\boxed{\begin{aligned} m_a \ddot{\eta}_a &= -(k_a + k_c)\eta_a + k_c \eta_b \\ m_b \ddot{\eta}_b &= + k_c \eta_a - (k_b + k_c)\eta_b \end{aligned}} \qquad (*)$$

coupled linear 2nd order o.d.e's.

Try <u>normal mode ansatz</u>: all parts oscillate w/ some frequency (not necessarily same amplitude or phase)

$$\begin{cases} \eta_a = f_a e^{i\omega t} \\ \eta_b = f_b e^{i\omega t} \end{cases}$$

f_a, f_b are complex (includes amp. + phase)
Take real part at the end.

$$-m_a \omega^2 f_a e^{i\omega t} = -(k_a + k_c) f_a e^{i\omega t} + k_c f_b e^{i\omega t}$$

$$-m_b \omega^2 f_b e^{i\omega t} = k_c f_a e^{i\omega t} - (k_b + k_c) f_b e^{i\omega t}$$

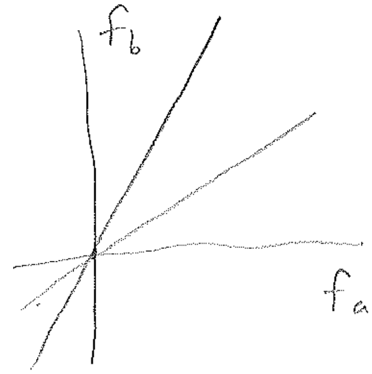
$e^{i\omega t}$'s cancel out

$$[m_a \omega^2 - k_a - k_c] f_a + k_c f_b = 0$$

$$k_c f_a + [m_b \omega^2 - k_b - k_c] f_b = 0$$

$$f_b = -\frac{(m_a\omega^2 - k_a - k_c)}{k_c} f_a$$

$$f_b = -\frac{k_c}{(m_b\omega^2 - k_b - k_c)} f_a$$



in general <u>only</u> solution is $f_a = f_b = 0$ (lines intersect at origin)

unless the equations are not independent (ie slopes equal)

$$\frac{m_a\omega^2 - k_a - k_c}{k_c} = \frac{k_c}{m_b\omega^2 - k_b - k_c}$$

$$(m_a\omega^2 - k_a - k_c)(m_b\omega^2 - k_b - k_c) = k_c^2 = 0$$

quadratic eqn for ω^2

$\Rightarrow$ two roots $\omega^{(1)2}$, $\omega^{(2)2}$

The positive square roots

$\omega^{(1)}$, $\omega^{(2)}$

are the normal mode frequencies

Start again w/ (*)

$$\ddot{\eta}_a = -\left(\frac{k_a+k_c}{m_a}\right)\eta_a + \frac{k_c}{m_a}\eta_b$$

$$\ddot{\eta}_b = \frac{k_c}{m_b}\eta_a - \left(\frac{k_b+k_c}{m_b}\right)\eta_b$$

Rewrite as matrix equation

$$\begin{pmatrix}\ddot{\eta}_a\\ \ddot{\eta}_b\end{pmatrix} = \underbrace{\begin{bmatrix} -\frac{(k_a+k_c)}{m_a} & \frac{k_c}{m_a} \\ \frac{k_c}{m_b} & -\frac{(k_b+k_c)}{m_b}\end{bmatrix}}_{\mathbb{M}} \begin{pmatrix}\eta_a\\ \eta_b\end{pmatrix}$$

Normal mode ansatz:

$$\begin{pmatrix}\eta_a\\ \eta_b\end{pmatrix} = \begin{pmatrix}f_a\\ f_b\end{pmatrix} e^{i\omega t} \quad \Rightarrow \begin{pmatrix}\ddot{\eta}_a\\ \ddot{\eta}_b\end{pmatrix} = -\omega^2 \begin{pmatrix}f_a\\ f_b\end{pmatrix} e^{i\omega t}$$

$e^{i\omega t}$ cancels out

$$\mathbb{M}\underbrace{\begin{pmatrix}f_a\\ f_b\end{pmatrix}}_{\text{eigenvector of } \mathbb{M}} = \underbrace{-\omega^2}_{\text{eigenvalue of } \mathbb{M}}\begin{pmatrix}f_a\\ f_b\end{pmatrix}$$

eigenvalue equation

$$= \begin{pmatrix}-\omega^2 & 0\\ 0 & -\omega^2\end{pmatrix}\begin{pmatrix}f_a\\ f_b\end{pmatrix}$$

$$= -\omega^2 \mathbb{1} \begin{pmatrix}f_a\\ f_b\end{pmatrix}$$

$$\boxed{\left(\mathbb{M} + \omega^2 \mathbb{1}\right)\begin{pmatrix} f_a \\ f_b \end{pmatrix} = 0}$$

unique solution is $\begin{pmatrix} f_a \\ f_b \end{pmatrix} = 0$ unless $\det\left(\mathbb{M} + \omega^2 \mathbb{1}\right) = 0$

$$\mathbb{M} + \omega^2 \mathbb{1} = \begin{bmatrix} -\frac{(k_a + k_c)}{m_a} + \omega^2 & \frac{k_c}{m_a} \\ \frac{k_c}{m_b} & -\frac{(k_b + k_c)}{m_b} + \omega^2 \end{bmatrix}$$

Recall $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC$

$$\det\left(\mathbb{M} + \omega^2 \mathbb{1}\right) = \left(\omega^2 - \frac{k_a + k_c}{m_a}\right)\left(\omega^2 - \frac{k_b + k_c}{m_b}\right) - \left(\frac{k_c}{m_a}\right)\left(\frac{k_c}{m_b}\right) = 0$$

quadratic equation for ω^2,
Can show that solutions are <u>real</u> + <u>positive</u>
Call the positive square roots $\omega^{(1)}$ and $\omega^{(2)}$

Simple case: $m_a = m_b = m$
$k_a = k_b = k \neq k_c$

$$\left(\omega^2 - \frac{k + k_c}{m}\right)^2 = \left(\frac{k_c}{m}\right)^2$$

$$\omega^2 - \left(\frac{k + k_c}{m}\right) = \pm \frac{k_c}{m}$$

$$\omega^2 = \frac{k}{m} + \frac{k_c}{m} \pm \frac{k_c}{m}$$

$$\omega^{(1)2} = \frac{k}{m} \quad \Rightarrow \quad \omega^{(1)} = \sqrt{\frac{k}{m}}$$

$$\omega^{(2)2} = \frac{k + 2k_c}{m} \quad \Rightarrow \quad \omega^{(2)} = \sqrt{\frac{k + 2k_c}{m}}$$

Background

$$0 = \begin{vmatrix} \omega^2 - (\alpha+\beta) & \beta \\ \gamma & \omega^2 - (\delta+\gamma) \end{vmatrix}$$

$\alpha = \frac{k_a}{m_a}$

$\beta = \frac{k_c}{m_a}$

$\gamma = \frac{k_c}{m_b}$

$\delta = \frac{k_b}{m_b}$

$$0 = [\omega^2 - (\alpha+\beta)][\omega^2 - (\delta+\gamma)] - \gamma\beta$$

$$= \omega^4 - (\alpha+\beta+\gamma+\delta)\,\omega^2 + \underbrace{(\alpha+\beta)(\gamma+\delta) - \gamma\beta}_{\alpha\gamma + \beta\gamma + \alpha\delta}$$

$$\omega^2 = \frac{1}{2}\left[(\alpha+\beta+\gamma+\delta) \pm \sqrt{\underbrace{(\alpha+\beta+\gamma+\delta)^2 - 4(\alpha\gamma+\beta\gamma+\alpha\delta)}_{\leq (\alpha+\beta+\gamma+\delta)^2}}\right]$$

$\therefore \; \omega^2 > 0$

(Also discriminant

$$= (\alpha+\beta+\gamma+\delta)^2 - 4(\alpha+\beta)(\gamma+\delta) + 4\gamma\beta$$

$$= (\alpha+\beta-\gamma-\delta)^2 + 4\beta\gamma > 0$$

so ω^2 are real)

Now find the eigenvector $\begin{pmatrix} f_a \\ f_b \end{pmatrix}$ for each normal mode

$$(\mathbb{M} + \omega^2 \mathbb{1}) \begin{pmatrix} f_a \\ f_b \end{pmatrix} = 0$$

$$\begin{pmatrix} \omega^2 - \frac{k+k_c}{m} & \frac{k_c}{m} \\ \frac{k_c}{m} & \omega^2 - \frac{k+k_c}{m} \end{pmatrix} \begin{pmatrix} f_a \\ f_b \end{pmatrix} = 0$$

$$\left(\omega^2 - \frac{(k+k_c)}{m}\right) f_a + \frac{k_c}{m} f_b = 0$$

First normal mode: $\omega^{(1)} = \sqrt{\frac{k}{m}}$

$$\Rightarrow \quad -\frac{k_c}{m} f_a^{(1)} + \frac{k_c}{m} f_b^{(1)} = 0$$

$$\Rightarrow \quad f_a^{(1)} = f_b^{(1)} \qquad \text{(symmetric)}$$

overall amplitude is arbitrary so choose $f_a^{(1)} = f_b^{(1)} = 1$

$$\begin{pmatrix} \eta_a^{(1)} \\ \eta_b^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k}{m}}t}$$

2nd normal mode: $\omega^{(2)} = \sqrt{\frac{k+2k_c}{m}}$

$$+\frac{k_c}{m} f_a^{(2)} + \frac{k_c}{m} f_b^{(2)} = 0$$

$$\Rightarrow \quad f_b^{(2)} = -f_a^{(2)} \qquad \text{(antisymmetric)}$$

choose $f_a^{(2)} = 1$, $f_b^{(2)} = -1$

$$\begin{pmatrix} \eta_a^{(2)} \\ \eta_b^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{k+2k_c}{m}}t}$$

Strategy for normal mode problems

- identify degrees of freedom ($\# = N$)

  let $\eta_a, \eta_b, \ldots$ = (small) displacements from eqbm

- identify forces acting on each d.o.f.

  Then $m_a \ddot{\eta}_a = F_a$ where F_a is linear in all the η's.

- assemble $\eta_a, \ldots$ into a vector $\eta = \begin{pmatrix} \eta_a \\ \eta_b \\ \vdots \end{pmatrix}$

  Then $\ddot{\eta} = \mathbb{M}\eta$ where $\mathbb{M}$ = (N×N) matrix constructed from the force

- normal mode ansatz

  $\eta = f e^{i\omega t}$ ie $\begin{pmatrix} \eta_a \\ \eta_b \\ \vdots \end{pmatrix} = \begin{pmatrix} f_a \\ f_b \\ \vdots \end{pmatrix} e^{i\omega t}$ ← same frequency for all parts

  $\ddot{\eta} = -\omega^2 f e^{i\omega t} = \mathbb{M} f e^{i\omega t}$

  $(\mathbb{M} + \omega^2 \mathbb{1}) f = 0$ eigenvalue equation

- Eigenvalue equation has nontrivial solutions iff

  $\det(\mathbb{M} + \omega^2 \mathbb{1}) = 0$

  (Nth order eqn for ω^2) $\Rightarrow$ find N roots $\Rightarrow \omega^{(i)}$, $i = 1, \ldots, N$

- For each $\omega^{(i)}$, solve $(\mathbb{M} + \omega^{(i)2} \mathbb{1}) f^{(i)} = 0$ for eigenvectors $f^{(i)}$

  $\Rightarrow$ normal mode solutions $\boxed{\eta^{(i)} = f^{(i)} e^{i\omega^{(i)} t}}$

- Write the most general solution

$$\eta(t) = \sum_{i=1}^{n} f^{(i)} \mathrm{Re}\left[c_i e^{i\omega^{(i)} t} \right] \quad \leftarrow \text{Form III}$$

$$= \sum_{i=1}^{n} f^{(i)} A_i \cos(\omega^{(i)} t + \phi^{(i)}) \quad \leftarrow \text{form I}$$

$$= \sum_{i=1}^{n} f^{(i)} \left[a_i \cos(\omega^{(i)} t) + b_i \sin(\omega^{(i)} t) \right] \quad \leftarrow \text{form II}$$

These have 2N arbitrary constants a_i, b_i, $(i = 1, \ldots N)$ just enough for 2N initial conditions, $\eta_a(0), \dot{\eta}_a(0), \eta_b(0), \dot{\eta}_b(0), \ldots$

a_i, b_i "normal mode coordinates"

- use initial conditions to solve for a_i, b_i. Done!

Special case: <u>zero mode</u>

If $\omega = 0$, the solutions above only give one of the two solutions

$$\eta = a \cos \omega t + b \sin \omega t \longrightarrow a$$

Need a second solution.

Recall the eqn: $\ddot{\eta} + \cancel{\omega^2} \eta = 0$

$$\eta = a + bt$$

~~or Taylor series $\eta = a(1 - \frac{1}{2}\omega^2 t^2 + \ldots) + b(\omega t - \frac{1}{3!}\omega^3 t^3 \ldots)$~~

~~$\longrightarrow a + (b\omega) t$~~, ~~$b'$~~

$$\ddot{z} = -g$$
$$z_p = -\tfrac{1}{2} g t^2$$
$$z_c = z_1 + v_0 t$$
$$z = z_p + z_c$$

$$\longrightarrow f^{(i)} \left[a_i + b_i t \right]$$

For our problem use form II

$$\begin{pmatrix} \eta_a \\ \eta_b \end{pmatrix} = \sum_{i=1}^{2} \begin{pmatrix} f_a^{(i)} \\ f_b^{(i)} \end{pmatrix} \left[a_i \cos \omega^{(i)} t + b_i \sin \omega^{(i)} t \right]$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[a_1 \cos \omega^{(1)} t + b_1 \sin \omega^{(1)} t \right] + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[a_2 \cos \omega^{(2)} t + b_2 \sin \omega^{(2)} t \right]$$

4 undetermined constants a_1, b_1, a_2, b_2

4 init. conditions $\begin{pmatrix} \eta_a(0) \\ \eta_b(0) \end{pmatrix}$ and $\begin{pmatrix} \dot\eta_a(0) \\ \dot\eta_b(0) \end{pmatrix}$

Example: both masses initially at rest.

m_b at equilibrium

m_a displaced δ from equilibrium

$$\begin{pmatrix} \eta_a(0) \\ \eta_b(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} a_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} a_2 = \begin{pmatrix} \delta \\ 0 \end{pmatrix}$$

$$\left. \begin{aligned} a_1 - a_2 &= 0 \\ a_1 + a_2 &= \delta \end{aligned} \right\} \Rightarrow a_1 = a_2 = \frac{\delta}{2}$$

$$\begin{pmatrix} \dot\eta_a(0) \\ \dot\eta_b(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \omega^{(1)} b_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \omega^{(2)} b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow b_1 = b_2 = 0$$

Solution:

$$\begin{pmatrix} \eta_a(t) \\ \eta_b(t) \end{pmatrix} = \frac{\delta}{2} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos \omega^{(1)} t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos \omega^{(2)} t \right]$$

check initial condition

Define $\omega = \frac{\omega_1 + \omega_2}{2}$

$\Delta = \frac{\omega_1 - \omega_2}{2}$

$$\Rightarrow \begin{cases} \omega_1 = \omega + \Delta \\ \omega_2 = \omega - \Delta \end{cases}$$

$$\cos \omega^{(1)} t = \cos \omega t \cos \Delta t - \sin \omega t \sin \Delta t$$

$$\cos \omega^{(2)} t = \cos \omega t \cos \Delta t + \sin \omega t \sin \Delta t$$

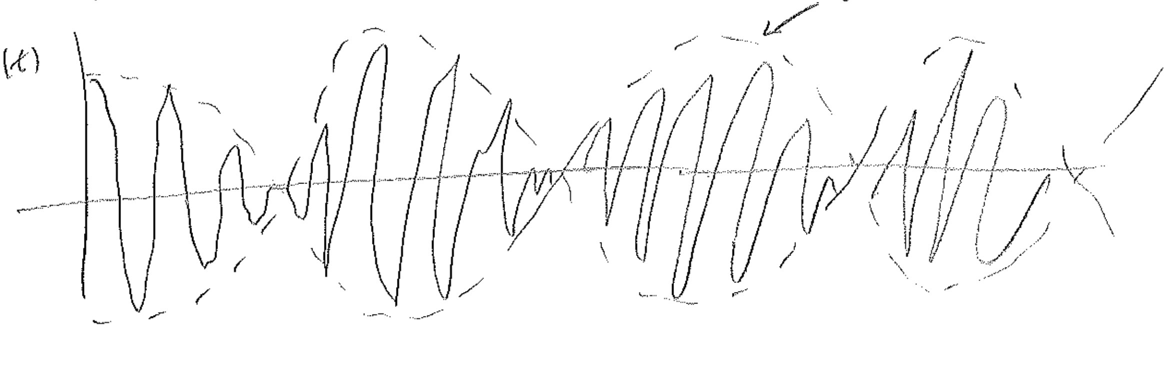
$$\Rightarrow \begin{pmatrix} \eta_a(t) \\ \eta_b(t) \end{pmatrix} = f \begin{pmatrix} \cos(\omega t) \cos(\Delta t) \\ \sin(\omega t) \sin(\Delta t) \end{pmatrix}$$

If coupling is weak then $k_c \ll k$ so $\omega^{(1)} \approx \omega^{(2)}$
and $\Delta \ll \omega$
$\cos(\Delta t)$ is slowly changing fn

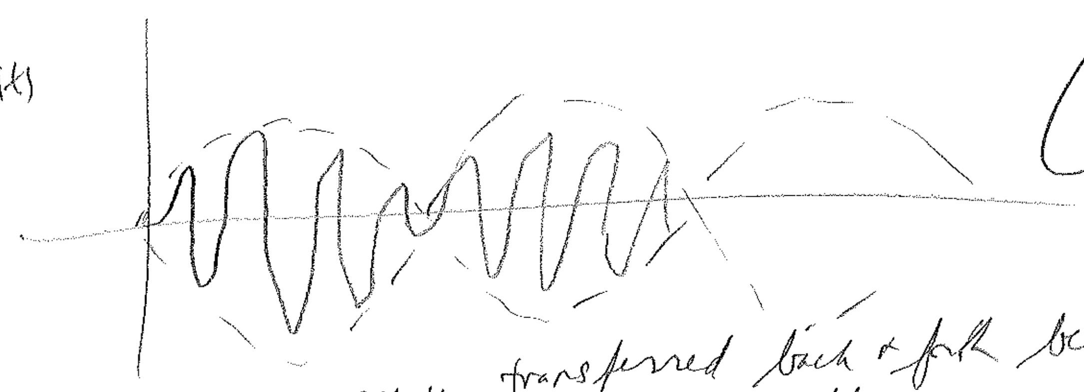
Slow function is envelope

envelope = $|\cos \Delta t|$

$\eta_a(t)$



$\eta_b(t)$



demo coupled pendula

energy transferred back & forth between the 2 oscillators