[return to harmonic oscillator]

F1

Recall $\ddot{x} + \omega_0^2 x = 0$

has solutions $\begin{cases} e^{i\omega_0 t} = \cos\omega_0 t + i \sin\omega_0 t \\ e^{-i\omega_0 t} = \cos\omega_0 t - i \sin\omega_0 t \end{cases}$

These are linearly independent. Since eqn is 2nd order the most general solution is

$$x = \frac{1}{2} c e^{i\omega_0 t} + \frac{1}{2} d e^{-i\omega_0 t}$$

where c, d are complex constants

However x must be real $\therefore$ $x = x^*$ so $d = c^*$

(physical displacement)

$$\Rightarrow x = \frac{1}{2} c e^{i\omega_0 t} + \frac{1}{2} c^* e^{-i\omega_0 t}$$

$$= \frac{1}{2}\left[\left(c e^{i\omega_0 t}\right) + \left(c e^{i\omega_0 t}\right)^*\right]$$

$$\boxed{x(t) = \mathrm{Re}\left(c e^{i\omega_0 t}\right)}$$ Form III

contains 2 arbitrary real constants,

$\Rightarrow \mathrm{Re}\, c, \mathrm{Im}\, c$

one can relate Form III to Form II

$$x(t) = \mathrm{Re}\left(c\, e^{i\omega_0 t} \right)$$

$$= \mathrm{Re}\left([\mathrm{Re}\, c + i\, \mathrm{Im}\, c][\cos\omega_0 t + i \sin\omega_0 t] \right)$$

$$= \underbrace{(\mathrm{Re}\, c)}_{a} \cos\omega_0 t + \underbrace{(-\mathrm{Im}\, c)}_{b} \sin\omega_0 t$$

$$\Rightarrow \begin{array}{l} a = \mathrm{Re}\, c \\ b = -\mathrm{Im}\, c \end{array} \Rightarrow c = a - ib$$

and to Form I

$$x(t) = A \cos(\omega_0 t + \phi)$$

$$= A\, \mathrm{Re}\, e^{i(\omega_0 t + \phi)}$$

$$= \mathrm{Re}\left(\underbrace{A e^{i\phi}}_{c} e^{i\omega_0 t} \right)$$

$$c = A e^{i\phi}$$

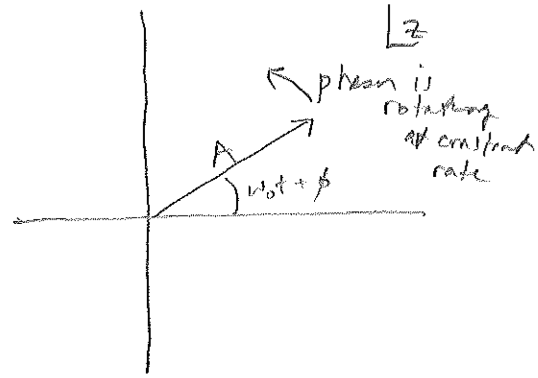
$$\begin{cases} A = |c| \\ \phi = \arctan\left(\dfrac{\mathrm{Im}\, c}{\mathrm{Re}\, c} \right) \end{cases}$$

Graphical representation

$$x(t) = \mathrm{Re}\,[\underbrace{A e^{i(\omega_0 t + \phi)}}_{z}]$$

"phasor"

project onto real axis



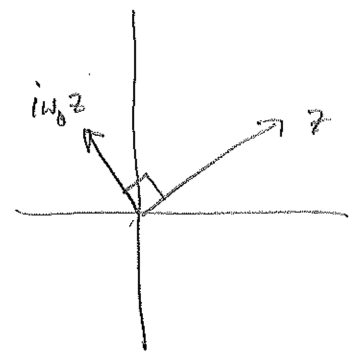
$$v(t) = \frac{dx}{dt} = \mathrm{Re}\,[\, i\omega_0 A\, e^{i(\omega_0 t + \phi)}]$$

$i = e^{\frac{i\pi}{2}}$

$$= \mathrm{Re}\,[\omega_0 A\, e^{i(\omega_0 t - \phi + \frac{\pi}{2})}]$$

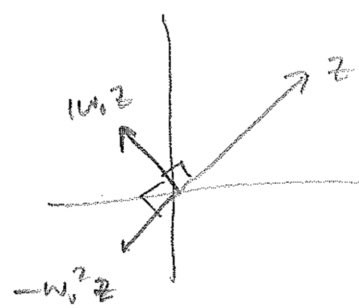
differentiation multiplies z by i + rotates phasor by 90°

v leads x by 90°



$$a(t) = \frac{dv}{dt} = \mathrm{Re}\,[\, i\omega_0^2 A\, e^{i(\omega_0 t - \phi + \frac{\pi}{2})}]$$

$$= \mathrm{Re}\,[\omega_0^2 A\, e^{i(\omega_0 t - \phi + \pi)}]$$



a leads v by 90°
and x by 180°

$$a = -\omega_0^2 x = -\frac{k}{m} x \quad \checkmark$$

Let L be a real n^{th} order differential operator

$$L = a_n(x)\frac{d^n}{dx^n} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

ie the coefficients $a_m(x)$ are real-valued functions.

<u>Principle of decomposition for a real linear homogeneous o.d.e.</u>

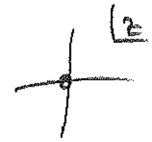
Suppose y is a complex solution of $Ly = 0$

[a linear o.de. can always have cx solns.
If y_1 and y_2 are real solutions, then $y_1 + i y_2$ is a soln]

Then $(\mathrm{Re}\, y)$ and $(\mathrm{Im}\, y)$ are seperately solutions.

Proof: $Ly = L(\mathrm{Re}\, y + i\, \mathrm{Im}\, y)$

$$= \underbrace{L(\mathrm{Re}\, y)}_{\substack{\text{real because} \\ L \text{ is real}}} + i\, \underbrace{L(\mathrm{Im}\, y)}_{\text{real}} = 0$$



Now if $z = 0$ then both $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ vanish.

$$\Rightarrow \quad L(\mathrm{Re}\, y) = 0$$
$$L(\mathrm{Im}\, y) = 0 \qquad \text{QED}$$

Thus, since $e^{i\omega_0 t}$ solves h.o. eqn, then so do $\cos \omega_0 t = \mathrm{Re}\, e^{i\omega_0 t}$ and $\sin \omega_0 t = \mathrm{Im}\, e^{i\omega_0 t}$