Complex numbers

$x - 1 = 0 \Rightarrow x = 1$

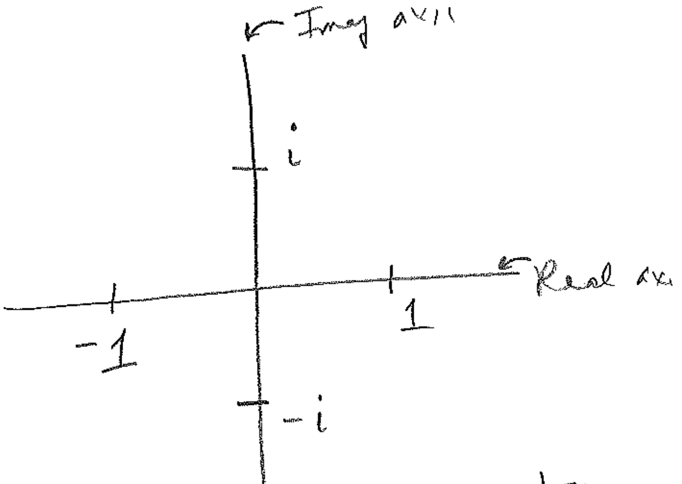
$x + 1 = 0 \Rightarrow x = -1$

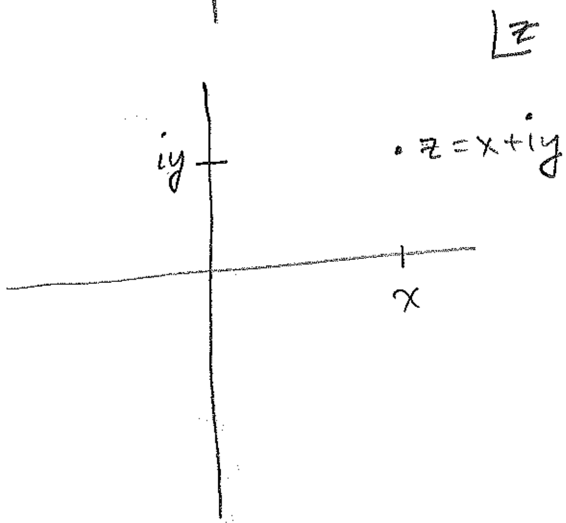
$x^2 - 1 = 0 \Rightarrow x = \pm 1$

$x^2 + 1 = 0 \Rightarrow x = \pm i$

$x^2 + 2x + 1 = 0 \Rightarrow$ complex

$x^n \pm 1 = 0 \Rightarrow$ still complex.



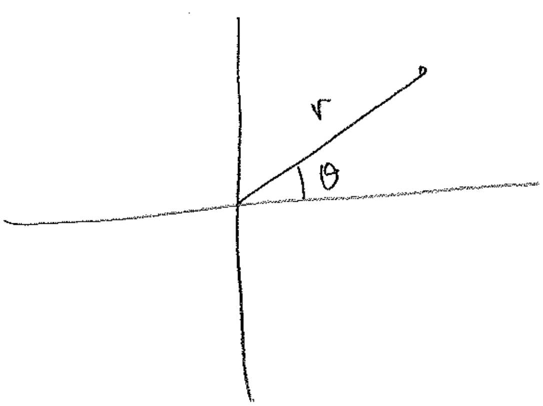


Let x, y be real numbers

$z = x + iy$ (Cartesian form)

$x = \text{Re}\, z$

$y = \text{Im}\, z$



$r = \sqrt{x^2+y^2}$ = magnitude $= |z|$

$\theta = \arctan\left(\frac{y}{x}\right)$ = phase

$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

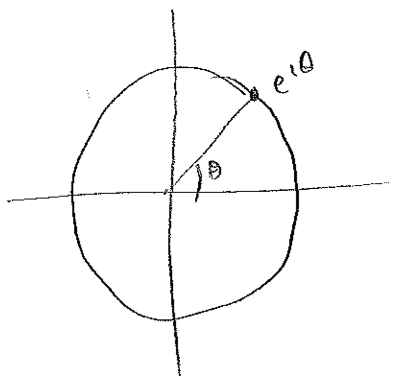
$z = re^{i\theta}$ (polar form)

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4$$

$$= 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!}$$

$$= \Big(\underbrace{1 - \tfrac{1}{2!}\theta^2 + \tfrac{1}{4!}\theta^4 + \ldots}_{\cos\theta}\Big) + i\Big(\underbrace{\theta - \tfrac{\theta^3}{3!} + \ldots}_{\sin\theta}\Big)$$

$e^{i\theta}$ is called a complex *phase*



$$e^{\frac{i\pi}{2}} = i$$

$$e^{i\pi} = -1$$

$$e^{\frac{3\pi i}{2}} = -i$$

$$e^{2\pi i} = 1$$

$$e^{2\pi i n} = 1 \quad \text{for } n \in \mathbb{Z}$$

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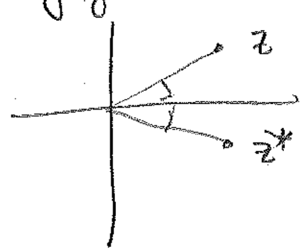
	z	z^* = complex conjugate
Cartesian	$= x + iy$	$= x - iy$
Polar	$= re^{i\theta}$	$= re^{-i\theta}$

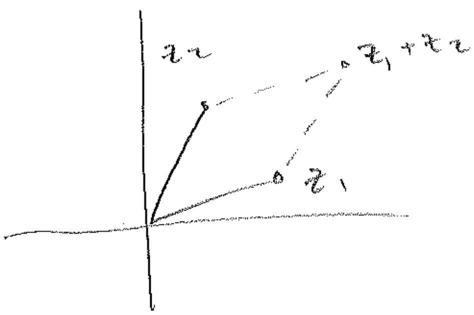


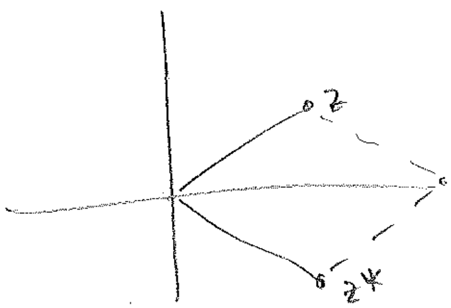
reflect across real axis

Addition is easy using Cartesian form.

$$z_1 + z_2 = (x_1 + i y_1) + (x_2 + i y_2) = (x_1 + x_2) + i(y_1 + y_2)$$

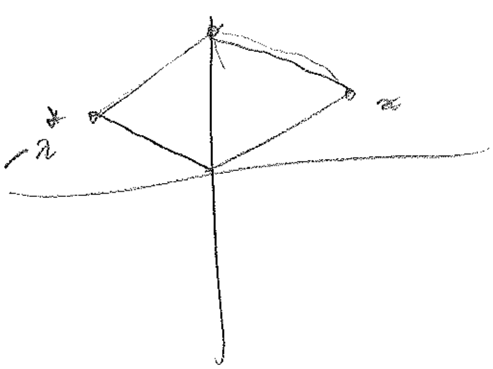
"vector addition in complex plane"





$$z + z^* = (x + iy) + (x - iy) = 2x$$

$$x = \mathrm{Re}\, z = \frac{1}{2}(z + z^*)$$



$$z - z^* = 2iy$$

$$y = \mathrm{Im}\, z = \frac{1}{2i}(z - z^*)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \mathrm{Re}\, e^{i\theta} = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

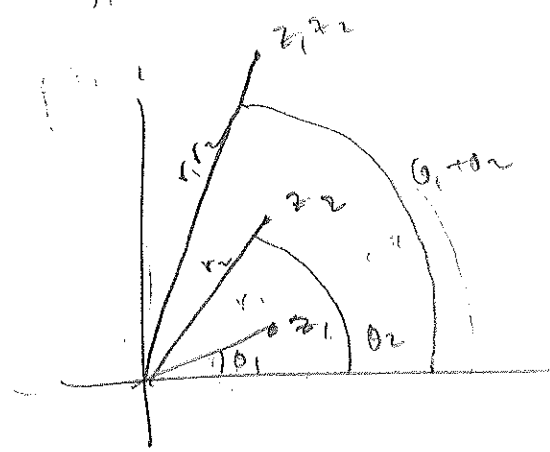
$$\sin\theta = \mathrm{Im}\, e^{i\theta} = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Multiplication

Cartesian

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$



Polar

$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2})$

$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$

- Multiply magnitudes
- Add phases

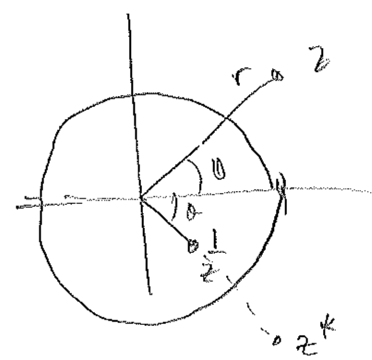
$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$

$z z^* = (re^{i\theta})(re^{-i\theta}) = r^2 = |z|^2$

Inverse

$\frac{1}{z} = \frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{x-iy}{(x^2+y^2)} = \left(\frac{x}{x^2+y^2}\right) - i\left(\frac{y}{x^2+y^2}\right)$

$\frac{1}{z} = \frac{1}{re^{i\theta}} = \left(\frac{1}{r}\right)e^{-i\theta}$



Also $\frac{1}{z} = \frac{z^*}{z z^*} = \frac{z^*}{|z|^2}$

Division

In cartesian form

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{|z_2|^2}$$ ← compute using cartesian

In polar form

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$$

- divide magnitude
- subtract phase

Note $$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

Easy way to derive/remember trig addition formulas

$$e^{i(a+b)} = e^{ia} e^{ib}$$

$$\cos(a+b) + i\sin(a+b) = (\cos a + i\sin a)(\cos b + i\sin b)$$

$$= (\cos a\cos b - \sin a\sin b) + i(\sin a\cos b + \cos a\sin b)$$

equate real + imag parts.

Powers :

z^{α} α real

$$z^{\alpha} = (re^{i\theta})^{\alpha} = r^{\alpha} e^{i(\alpha\theta)}$$ { magnitude raised to α ; phase multiplied by α

$$(e^{i\theta})^n = e^{in\theta}$$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin n\theta$$ de Moivre's thm

eg $n = 2$

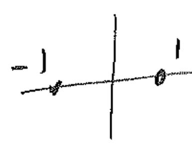
$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) = \cos(2\theta) + i\sin(2\theta)$$

$$(\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$$

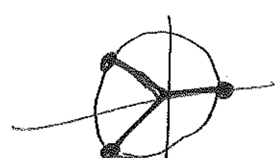
Roots

$$\sqrt[\alpha]{z} = z^{\frac{1}{\alpha}} = r^{\frac{1}{\alpha}} e^{i\left(\frac{\theta}{\alpha}\right)}$$ { αth root of magnitude ; Divide phase by α

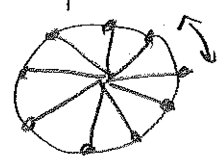
$$\sqrt[2]{e^{2\pi i}} = e^{\pi i} = -1$$



$$\sqrt[2]{1} = 1, -1$$



$$\sqrt[3]{1} = 1,\ e^{2\pi i/3},\ e^{4\pi i/3}$$



$\frac{2\pi}{n}$

$$\sqrt[n]{1} = \sqrt[n]{e^{2\pi i m}} = e^{2\pi i\left(\frac{m}{n}\right)}$$

[If asked about complex exponents: $z^{\alpha+i\beta} = (re^{i\theta})^{\alpha+i\beta} = e^{(\ln r + i\theta)(\alpha+i\beta)}$

Eric's fav

$$= e^{\alpha\ln r - \beta\theta}\, e^{i(\theta\alpha - \beta\ln r)}$$

$$= r^{\alpha} e^{i\alpha\theta} \cdot e^{-\beta\theta - i\beta\ln r}$$