Second order o.d.e

Newton's 2nd Law $\vec{F} = m\vec{a}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}}{m} \quad (3 \text{ eqns})$$

$\left\{ \begin{array}{l} \vec{r} = (x, y, z) \\ \quad = \text{dependent variable} \\ t = \text{indep. variable} \end{array} \right.$

$$\frac{d^2x}{dt^2} = \frac{F_x}{m}, \quad \frac{d^2y}{dt^2} = \frac{F_y}{m}, \quad \frac{d^2z}{dt^2} = \frac{F_z}{m}$$

In general (unlike 1st order o.d.e's), 2nd order odes can't be solved by direct integration except in some special cases, namely

(a) one dimensional problems where force depends only on velocity but not position or time

(b) conservative forces

(A) one dimensional problem where force is independent of position and time

$$a = \frac{dv}{dt} = \frac{F(v)}{m} \qquad \text{[Separable]}$$

$$m\frac{dv}{F(v)} = dt$$

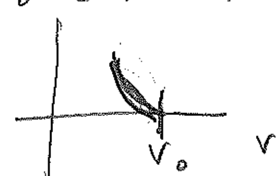
Integrate both sides w/ init. cond $v = v_0$ at $t = 0$

$$\underbrace{\int_{v_0}^{v} m\frac{dv}{F(v)}}_{} = \int_0^t dt$$

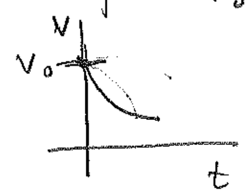
call the integral $G(v)$

$$G(v) - G(v_0) = t \qquad \Rightarrow t \text{ as a function of } v$$

$t = G(v) - G(v_0)$



Graphically invert



Invert this analytically if possible

$$v = H(t)$$

Then $\dfrac{dx}{dt} = H(t)$ separable

$$dx = H(t)\,dt$$

Integrate w/ init. cond $x = x_0$ at $t = 0$

$$\int_{x_0}^{x} dx = \int_0^t H(t)\,dt$$

$$x - x_0 = \int_0^t H(t)\,dt$$

[exercise: $F(v) = -bv$]

(2) A conservative force is one that obeys

$$\vec{F} = -\underset{\substack{\uparrow \\ \text{gradient}}}{\vec{\nabla}} U = \left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right)$$

[equiv. to path independent work, cf later]

for some continuous function $U(x, y, z)$
(Later we'll see that $U =$ potential energy)

$$m\vec{a} = \vec{F}$$

$$m\frac{d\vec{v}}{dt} = -\vec{\nabla} U$$

Trick: multiply both side by $\vec{v} = \frac{d\vec{r}}{dt}$

$$m\,\vec{v}\cdot\frac{d\vec{v}}{dt} = -\vec{\nabla} U \cdot \frac{d\vec{r}}{dt}$$

$$m\left(v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt} + v_z\frac{dv_z}{dt}\right) = -\left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt}\right)$$

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2 + \tfrac{1}{2}mv_z^2}_{K}\Big) = -\frac{dU}{dt} \qquad \text{(chain rule)}$$

$$\boxed{\begin{array}{c} U(x(t), y(t), z(t)) \\ \swarrow \quad \downarrow \quad \searrow \\ x(t) \quad y(t) \quad z(t) \\ \searrow \quad \downarrow \quad \swarrow \\ t \end{array}}$$

$$\frac{d}{dt}(K + U) = 0$$

$$K + U = \text{const, call it } E \leftarrow \text{use init. cond. to determine}$$

Energy conservation determines the magnitude (not direction!) of velocity in terms of position

$$\tfrac{1}{2}mv^2 = E - U(\vec{r})$$

$$|\vec{v}| = \sqrt{\frac{2}{m}\left(E - U(\vec{r})\right)}$$

For motion constrained to one dimension, conservation of energy determines velocity up to a sign (depends on initial conditions)

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

[separable]

$$\pm\frac{dx}{\sqrt{E-U(x)}} = \sqrt{\frac{2}{m}}\,dt$$

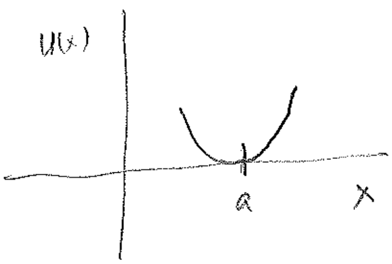
$$\pm\int_{x_0}^{x}\frac{dx}{\sqrt{E-U(x)}} = \sqrt{\frac{2}{m}}\int_0^t dt = \sqrt{\frac{2}{m}}\,t$$

use physical situation to determine the sign

integrate & invert if possible to find $x(t)$

Consider small perturbations about a stable equilibrium point $x = a$

$$U(x) = \underbrace{U(a)}_{\substack{\text{arbitrary,} \\ \text{set it to zero}}} + \underbrace{\frac{dU}{dx}(a)}_{= -F_x(a) = 0}(x-a) + \frac{1}{2}\underbrace{\frac{d^2U}{dx^2}(a)}_{\text{call this constant } k}(x-a)^2 + \ldots$$

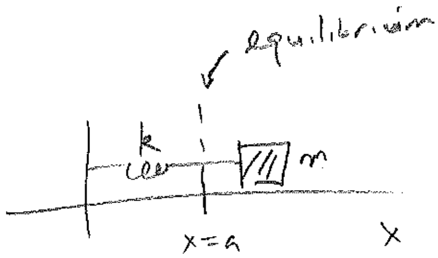


$$U(x) \approx \frac{1}{2}k(x-a)^2$$

$$F_x = -\frac{\partial U}{\partial x} \approx -k(x-a)$$

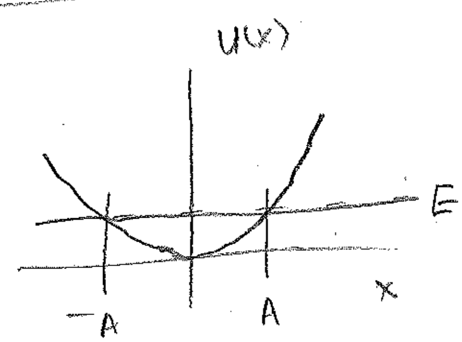
Hooke's law for a spring

(linear restoring force)



For simplicity, choose origin to be equilibrium pt.

$a = 0, \Rightarrow F_x = -kx \Rightarrow U = \frac{1}{2}kx^2$



$K + U = E$

max displacement $= A$, amplitude

At $x = A$, $K = 0$ and $U = \frac{1}{2}kA^2$

so $E = \frac{1}{2}kA^2$

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{k}{m}(A^2-x^2)}$$

choose + sign if moving to the right at time t.

$$\frac{dx}{\sqrt{A^2-x^2}} = \sqrt{\frac{k}{m}}\,dt$$

Trig substitution $x = A\sin\theta$
$dx = A\cos\theta\,d\theta$

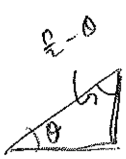
$$\frac{dx}{\sqrt{A^2-x^2}} = \frac{A\cos\theta\,d\theta}{\sqrt{A^2-A^2\sin^2\theta}} = d\theta = \sqrt{\frac{k}{m}}\,dt$$

$$\theta = \sqrt{\frac{k}{m}}\,t + C_2$$

$$\Rightarrow \boxed{x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t + C_2\right)}$$ harmonic oscillator

Define $\omega_0 \equiv \sqrt{\frac{k}{m}}$ = natural frequency

A, C_2 are 2 arbitrary constants (2nd order ode)
determined by initial conditions

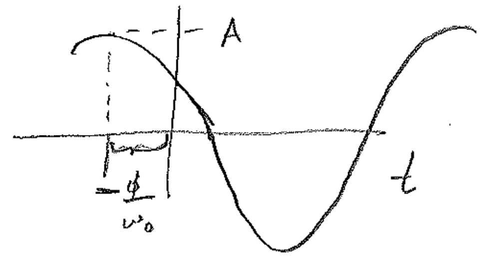


$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right) = \cos\left(\theta-\frac{\pi}{2}\right)$ so can also write

$$\boxed{x(t) = A\cos(\omega_0 t + \phi)}$$ where $\phi = C_2 - \frac{\pi}{2}$

↑
form I of harmonic oscillator soln

$\begin{cases} A = \text{amplitude} \\ \phi = \text{phase} \end{cases}$



Mass on a spring

$$F_x = m a_x$$

$$-kx = m \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0}$$ harmonic oscillator equation

This is a linear 2nd order o.d.e.

Let $Ly = 0$ be an nth order linear (homogeneous) o.d.e

Superposition principle
- If y_1 and y_2 are solutions, so is any linear combination $c_1 y(x) + c_2 y_2(x)$ $(c_1, c_2 = \text{const})$

  Proof: $L(c_1 y_1 + c_2 y_2) = c_1 \underbrace{(Ly_1)}_{0} + c_2 \underbrace{(Ly_2)}_{0} = 0$.

- $Ly = 0$ possesses exactly n linearly indep. solutions $y_1, y_2, \ldots, y_n$
- Linearly independent means cannot write any of them as a linear combination of the others

- General solution: the most general solution of $Ly = 0$ is $c_1 y_1(x) + c_2 y_2(x) \ldots + c_n y_n(x)$ where $c_1, \ldots, c_n$ are constants undetermined by the equation but are determined by initial or boundary conditions

[solution by superposition; general because n consts]

Harmonic oscillator has 2 linearly independent solutions

$\boxed{x(t) = A\cos(\omega_0 t + \phi)}$ Form I of solution

Set $\phi = 0 \Rightarrow$ $x_1(t) = \cos(\omega_0 t)$
Set $\phi = -\frac{\pi}{2} \Rightarrow$ $x_2(t) = \sin(\omega_0 t)$ } linearly independent because $\cos(\omega_0 t) \neq (\text{const}) \sin(\omega_0 t)$

Any other solution is a linear combination of these

$$x(t) = A\cos(\omega_0 t + \phi) = \underbrace{(A\cos\phi)}_{a}\cos\omega_0 t + \underbrace{(-A\sin\phi)}_{b}\sin\omega_0 t$$

$\boxed{x(t) = a\cos\omega_0 t + b\sin\omega_0 t}$ ← call this form II of solution

$$a = A\cos\phi \qquad \Rightarrow \qquad A = \sqrt{a^2 + b^2}$$
$$b = -A\sin\phi \qquad\qquad \tan\phi = -\frac{b}{a}$$

a, b are undetermined by the eqn; use initial conditions

$x(0) = a$
$v(t) = -a\omega_0 \sin\omega_0 t + b\omega_0\cos\omega_0 t$
$v(0) = b\omega_0$

$$\Rightarrow \quad x(t) = x(0)\cos\omega_0 t + \frac{v(0)}{\omega_0}\sin\omega_0 t$$

$$\ddot{x} + \omega_0^2 x = 0 \qquad \left[\ddot{x} = \frac{d^2x}{dt^2} \right]$$

linear ode's can sometimes be solved w/ an
exponential ansatz

Try $x(t) = e^{pt}$ where $p =$ const

$\ddot{x} = p^2 e^{pt}$

$\ddot{x} + \omega_0^2 x = (p^2 + \omega_0^2) x = 0$

either $x = 0$ (no solution) or $p^2 + \omega_0^2 = 0$

$p^2 = -\omega_0^2$

$p = \pm i\omega_0$

$x(t) = \underbrace{e^{i\omega_0 t}, \; e^{-i\omega_0 t}}$

2 linearly independent solutions

Recall Euler formula: $e^{i\omega_0 t} = \cos\omega_0 t + i \sin\omega_0 t$

$e^{-i\omega_0 t} = \cos\omega_0 t - i \sin\omega_0 t$

Same solutions as before