Linear ODEs

A <u>linear</u> differential equation is one in which each term contains at most one factor of the <u>dependent</u> variable or its derivatives

$$\underbrace{a_n(x)\frac{d^n y}{dx^n} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)\,y}_{Ly} = S(x) \qquad (*)$$

source term

observe: $L(cy) = c\,Ly$

$L(y_1+y_2) = Ly_1 + Ly_2$

$\therefore$ L = linear operator

n = order of the eqn

The most general soln of $(*)$ contains n undetermined constants (undetermined by the eqn)

simple e.g. $\frac{d^n y}{dx^n} = S(x)$

[integrate n times
n constants of integration]

A homogeneous linear differential eqn has $S(x) = 0$.

[i.e. contains exactly one factor of y or its derivatives in each term.]

$$Ly = 0$$

e.g. $\frac{dN}{dt} + \lambda N = 0$ w/ $a_1 = 1$, $a_2 = \lambda$

If y is a solution of a homogeneous linear ODE, then so is any constant multiple cy

(due to linearity)

Conversely since cy solves the equation for any c, the eqn ~~does~~ cannot determine the value of c. It is one of the undetermined constants.

e.g. for $n=1$, $N = N_0 e^{-\lambda t}$ [need init cond.]

N_0 undetermined constant

For an inhomogeneous linear ODE (ie $S(x) \neq 0$)

$$Ly = S(x) \qquad (*)$$

we call

$$Ly = 0 \qquad (**)$$

the associated homogeneous eq.

Let

- $y_p(x)$ be some solution to the inhomogeneous eqn $(*)$ (called "particular" solution) — it need not satisfy boundary cond.
- $y_c(x)$ be the most general solution to the associated eqn $(**)$ (called "complementary" solution) — it has n undetermined consts

Then

- $y_p(x) + y_c(x)$ is the most general solution of inhomog eqn $(*)$
  - soln because $L(y_p + y_c) = Ly_p + Ly_c = S + 0 = S$
  - most general because contains n undetermined consts
  - boundary conditions determine these coeffs

There is no universal recipe for $y_p(x)$. Start w/ trial solution w/ same form as $S(x)$ & modify as necessary

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If there are several different source terms, $S_1(x) + S_2(x)$ find a particular soln for each $Ly_{p1} = S_1$ and $Ly_{p2} = S_2$

Then $y = y_{p1} + y_{p2} + y_c$ is the most general solution

Radioactive series

Let radioactive species X_1 decay to daughter species X_2 w/ decay constant λ_1 [e.g. U → Th]

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad \Rightarrow \quad N_1(t) = \underbrace{N_1(0)}_{\text{fixed by initial conditions}} e^{-\lambda_1 t}$$

Let daughter species X_2 decay w/ decay const λ_2 [e.g. Th → Pa]

$$\frac{dN_2}{dt} = \underbrace{-\lambda_2 N_2}_{\substack{\text{rate at which}\\ X_2 \text{ decays}}} + \underbrace{\lambda_1 N_1}_{\substack{\text{rate at which}\\ X_1 \text{ decays into } X_2}}$$

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \underbrace{\lambda_1 N_1(0)\, e^{-\lambda_1 t}}_{\text{known source}}$$

associated eqn

$$\frac{dN_2}{dt} + \lambda_2 N_2 = 0 \quad \Rightarrow \text{complementary soln} \quad N_{2c} = \underbrace{A}_{\text{undetermined}} e^{-\lambda_2 t}$$

particular soln? [Try one w/ same form as source]

Try $N_{2p} = B e^{-\lambda_1 t}$

$$-\lambda_1 B e^{-\lambda_1 t} + \lambda_2 B e^{-\lambda_1 t} = \lambda_1 N_1(0) e^{-\lambda_1 t} \quad ✓$$

$$B = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0)$$

This works, provided $\lambda_1 \neq \lambda_2$. Otherwise need be more creative.

general solution

$$N_2 = N_{2c} + N_{2p}$$

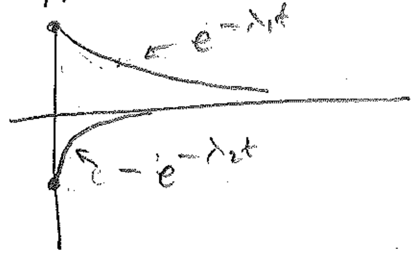
$$= A e^{-\lambda_2 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0) e^{-\lambda_1 t}$$

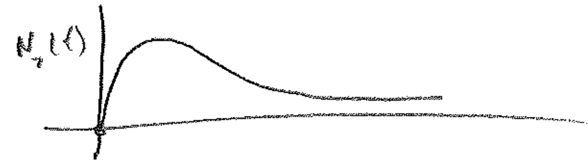
Suppose no daughter species present initially

$$N_2(0) = 0 \quad \Rightarrow \quad A = \frac{-\lambda_1}{\lambda_2 - \lambda_1} N_1(0)$$

$$N_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0) \left[e^{-\lambda_1 t} - e^{-\lambda_2 t} \right]$$

Suppose daughter decays more quickly than parent $(\lambda_2 > \lambda_1)$





2nd term dies away faster so $N_2(t) \xrightarrow[t \text{ large}]{} \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0) e^{-\lambda_1 t}$

Consider ratio of daughter to parent

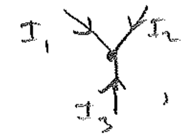
$$\frac{N_2(t)}{N_1(t)} \longrightarrow \frac{\lambda_1}{\lambda_2 - \lambda_1} = \frac{\tau_2}{\tau_1 - \tau_2} \xrightarrow[\text{if } \tau_1 \gg \tau_2]{} \frac{\tau_2}{\tau_1}$$

ratio → ratio of mean lives

"secular equilibrium"

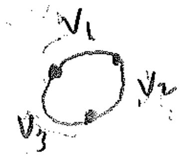
Circuits

Kirchoff' current law: at any junction (KCL)



, $\Sigma I_i = 0$ (charge conservation)

Kirchoff' voltage law: around any closed loop

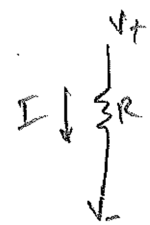


, $\Sigma V_i = 0$ (energy conservation)

(Caveat: no changing flux through loop, otherwise EMF)

Resistors

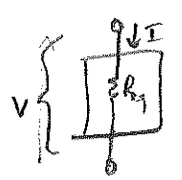
$V = IR$



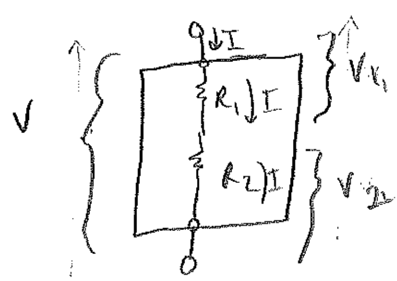
$V_+ - V_- = IR$

Series

Black box



$V = IR_{eq}$ so $R_{eq} \equiv \frac{V}{I}$



$I = I_1 = I_2$ by KCL

$V = V_1 + V_2$ by KVL

$= IR_1 + IR_2$

$= (R_1 + R_2)I$

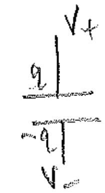
$R_{eq} = \frac{V}{I} = R_1 + R_2$

[exercise: do

$\Rightarrow$ KVL : $V_1 = V_2 = V$

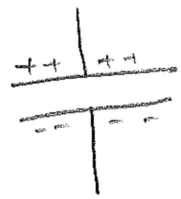
KCL : $I = I_1 + I_2 = \frac{V}{R_1} + \frac{V}{R_2} \Rightarrow R_{eq} = \frac{V}{I} = \frac{1}{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$]

Capacitance



$V_+ - V_- = \frac{q}{C}$

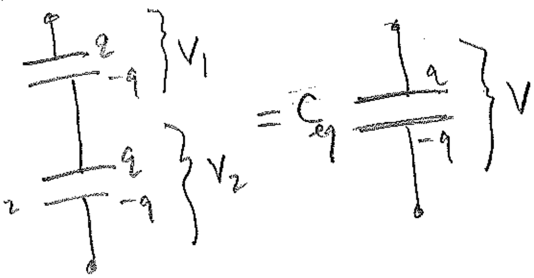
capacitor



$q = CV$

Capacitance $C = \frac{q}{V} = \frac{\text{Coul}}{\text{volts}} = \text{farad}$

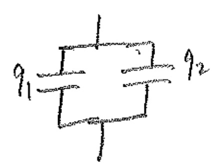
Add capacitors in series



$V = V_1 + V_2 = \frac{q}{C_1} + \frac{q}{C_2}$

$C_{eq} = \frac{q}{V} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2}$

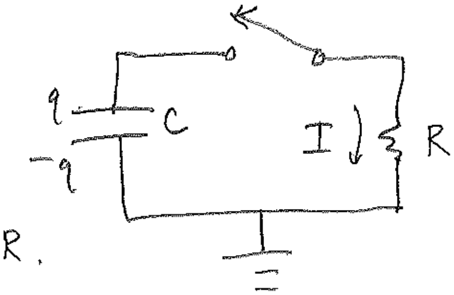
Exercise: show capacitors in parallel obey $C_{eq} = C_1 + C_2$

[

$q = q_1 + q_2 = C_1 V + C_2 V = (C_1 + C_2) V$]

Capacitor discharge

RC circuit:



Close switch at $t=0$.
Current flows down thru R.

observe $I = -\frac{dq}{dt}$

After $t>0$, KVL $\Rightarrow$ $V_R = V_c$

$$IR = \frac{q}{C}$$

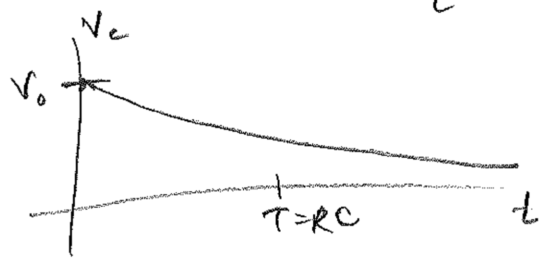
$$\frac{dq}{dt} = -\frac{q}{RC}$$

$$\frac{dV_c}{dt} = -\frac{V_c}{RC}$$

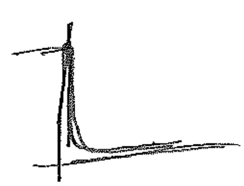
solve $\Rightarrow$ $V_c = A e^{-\frac{t}{\tau}}$ where $\tau = RC$

Initial condition: capacitor charged to V_0 at $t=0$ $\Rightarrow$ $A = V_0$

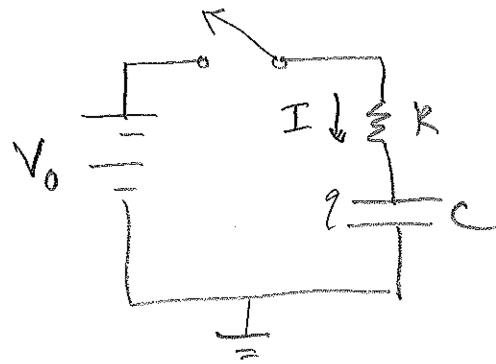
$$V_c = V_0 e^{-t/\tau}$$



Voltage across a capacitor cannot change discontinuously (unless current thru it is infinite)
(since $V_c = \frac{q}{C}$ / $\frac{dV_c}{dt} = \infty \Rightarrow I = \infty$)
only possible if $R=0$



Charging a capacitor



After close switch

$$I = \frac{dq}{dt}$$

$$\text{KVL} \Rightarrow V_R + V_C = V_0$$

$$IR + \frac{q}{C} = V_0$$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}$$

$$\frac{dV_c}{dt} + \frac{V_c}{RC} = \frac{V_0}{RC}$$

Inhomogeneous ode

two ways to proceed

because source term is constant we can solve it like this

$$\frac{dV_c}{dt} = -\frac{(V_c - V_0)}{RC}$$

$$\frac{dV_c}{V_c - V_0} = -\frac{dt}{RC}$$

$$\ln(V_c - V_0) = -\frac{t}{RC} + B$$

$$V_c - V_0 = A e^{-t/RC}$$

alternatively

Complementary solution

$$V_c = A e^{-t/RC}$$

particular solution: try const

$$V_c = V_0$$

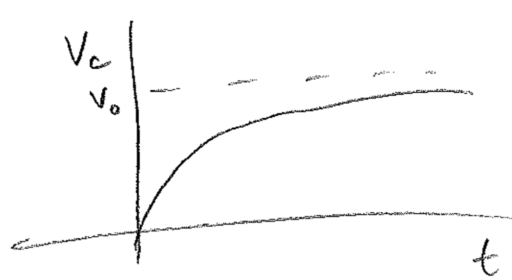
general solution:

$$V_c = V_0 + A e^{-t/RC}$$

same!

Initial condition: uncharged capacitor $V_c(0) = 0 \Rightarrow A = -V_0$

$$V_c = V_0(1 - e^{-t/RC})$$



Inductance

Inductor 

inductors oppose the change of current through them whereas resistors resist the current.

voltage across resistor required to produce current
voltage across inductor required to cause change of curr.

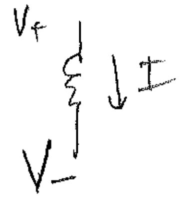
mechanical analogy:
resistance ~ friction (viscosity)
force required to make object move
inductance ~ mass (inertia)
force required to make object accelerate

inductors create magnetic fields: inertia of $\vec{B}$

Voltage drop across inductor causes current through it to change

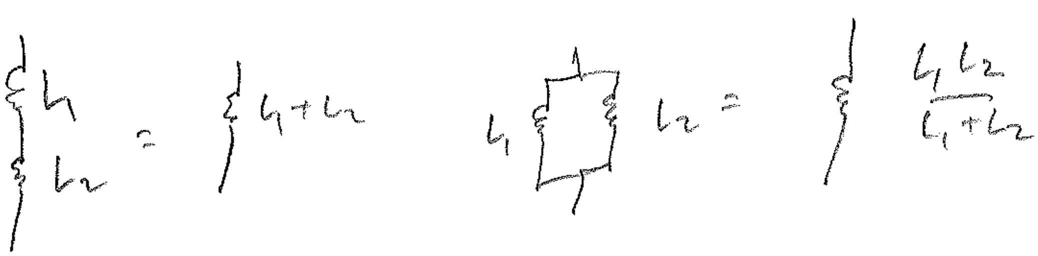
$$\frac{dI}{dt} = \frac{\Delta V_L}{L}$$

L = inductance units = $\frac{\text{volts}}{\left(\frac{\text{amps}}{\text{sec}}\right)}$ = henry



$$V_+ - V_- = L\,\frac{dI}{dt}$$

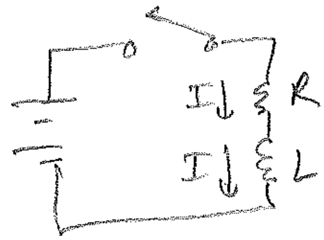
current through an inductor cannot change discontinuously (unless $\Delta V_L = \infty$)



L_1, L_2 = $L_1 + L_2$ L_1 L_2 = $\frac{L_1 L_2}{L_1 + L_2}$

[exercise: prove this]

RL circuit



- Initially no current through inductor
- close switch at $t = 0$

$$V_L + V_R = V_0$$

$$L \frac{dI}{dt} + IR = V_0$$

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{V_0}{L}$$

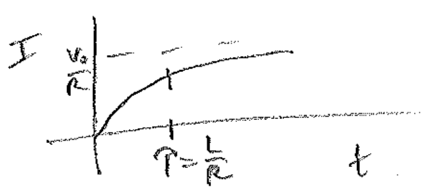
associated eqn $\frac{dI}{dt} + \frac{R}{L} I = 0$

complementary solution $I = A e^{-\frac{R}{L}t}$

particular solution $I = \frac{V_0}{R}$

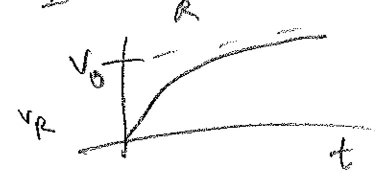
general $I = \frac{V_0}{R} + A e^{-\frac{R}{L}t}$

initially $I(0) = 0 \Rightarrow I(t) = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

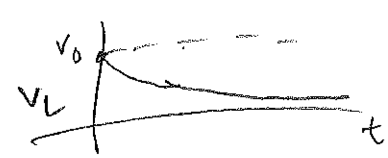


$\tau = \frac{L}{R} = \frac{\text{henry}}{\text{ohm}} = \text{sec}$ ✓

As $t \to \infty$, $I \to \frac{V_0}{R}$



$$V_R = IR = V_0\left(1 - e^{-\frac{R}{L}t}\right)$$



$$V_L = V_0 - V_R = V_0 e^{-\frac{R}{L}t}$$

or $L \frac{dI}{dt}$