First order ordinary differential equations

only first derivatives (First order)

only one independent variable (usually x or t) (ordinary)

① Exponential decay

Radioactivity = decay of atomic nuclei

Key experimental fact: the probability that a nucleus will decay is independent of how long it has existed

Let $N(t)$ = # of radioactive nuclei present in a sample at time t

Define decay constant λ as probability per unit time that any one nucleus will decay (called constant because indep of time)

Define activity $A(t)$ as number of nuclei in sample that decay per unit time [what you actually measure]

$$A(t) = \lambda N(t)$$

This reduces the number of nuclei present

$$\frac{dN}{dt} = -A = -\lambda N$$

$$\boxed{\frac{dN}{dt} + \lambda N = 0}$$ decay equation

A separable 1st order o.de [is one that] can be rearranged so [that] all occurrences of the dependent variable appear on one side of the eqn, and all occurrences of the independent variable appear on the other. Solve by integrating both sides.

$$\frac{dN}{dt} = -\lambda N$$

$$\frac{dN}{N} = -\lambda\, dt$$

const of integration

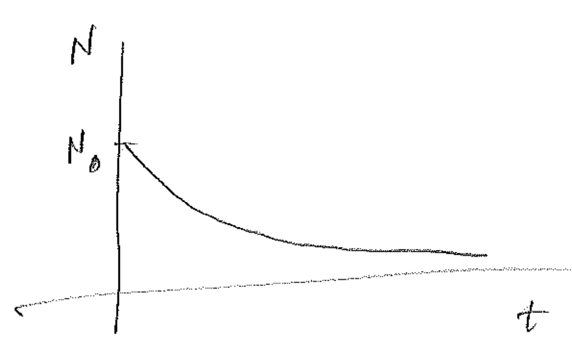
$$\ln N = -\lambda t + C_1$$

The solution of a 1st order eqn always contains one constant undetermined by the eqn.
It can be determined using initial (or boundary) conditions

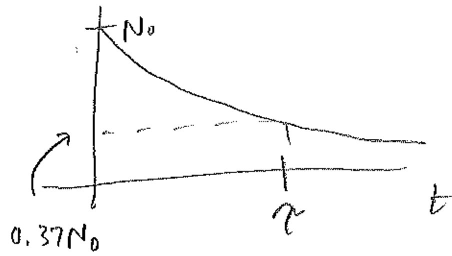
Let N_0 = # nuclei at $t=0$ $\Rightarrow$ $C_1 = \ln N_0$

$$\underbrace{\ln N - \ln N_0}_{\ln\left(\frac{N}{N_0}\right)} = -\lambda t$$

$$N(t) = N_0 e^{-\lambda t}$$



Define $\tau = \frac{1}{\lambda} \Rightarrow N(t) = N_0 e^{-\frac{t}{\tau}}$



$N(\tau) = N_0 e^{-1} \approx 0.37 N_0$

so after time τ only 37% of nuclei remain

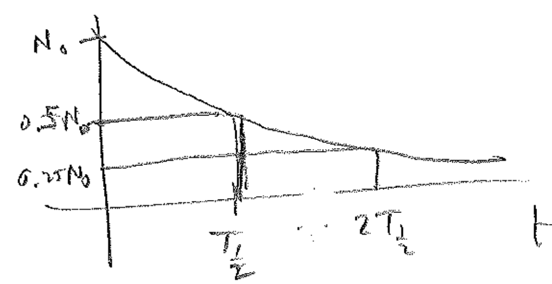
(one can show τ = mean life of a nucleus)

→ (see how to plot in mathematica)

Define half-life $T_{\frac{1}{2}}$ by $N(T_{\frac{1}{2}}) = \frac{1}{2} N_0$

(time it takes half of nuclei to decay)

Since $N(T_{\frac{1}{2}}) = N_0 e^{-T_{\frac{1}{2}}/\tau} \Rightarrow T_{\frac{1}{2}} = \tau \ln 2 \approx 0.69\tau$



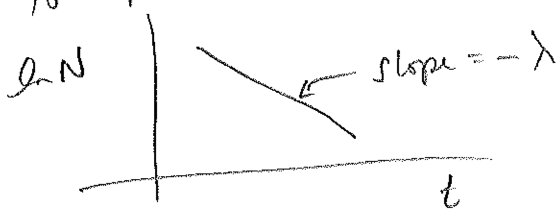
Prove to yourself that

$$N = N_0 \cdot \left(\frac{1}{2}\right)^{\frac{t}{T_{1/2}}}$$

so that for each half-life, only half of nuclei remain

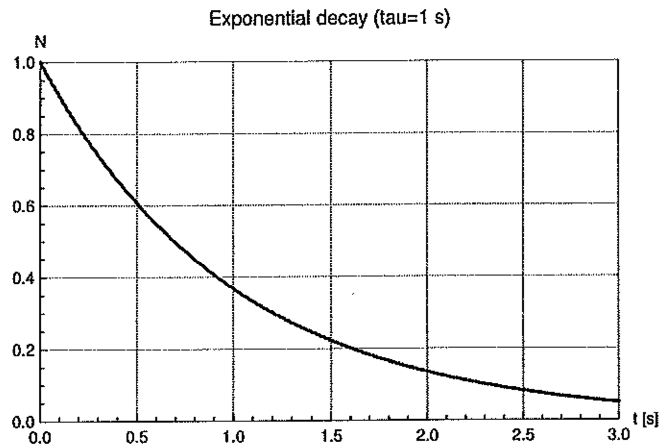
To measure τ, do a semilog plot of data

$\ln N = \ln N_0 - \lambda t$



In[5]:=
```
Plot[ Exp[-x], {x, 0, 3},
      PlotRange → { {0, 3}, {0, 1.0} },
      GridLines → Automatic,
      AxesLabel → {"t [s]", "N"},
      PlotLabel → "Exponential decay (tau=1 s)"]
```

Out[5]=



In[6]:=
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Export["ExpDecay.pdf", %]
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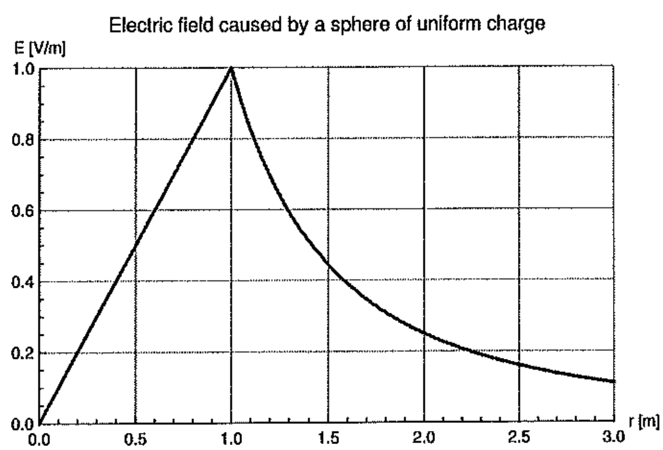
Out[6]= ExpDecay.pdf

In[7]:=
```
Plot[Piecewise[{{r, r < 1}, {1/r^2, r > 1}}], {r, 0, 3}, PlotRange → { {0, 3}, {0, 1} },
  GridLines → Automatic,
  AxesLabel → {"r [m]", "E [V/m]"},
  PlotLabel → "Electric field caused by a sphere of uniform charge"]
```

Out[7]=



In[8]:=
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Export["ElectricField.pdf", %]
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Out[8]= ElectricField.pdf

1
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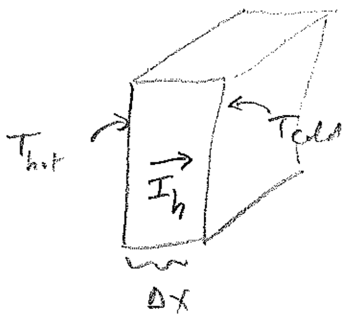
2. Thermal conduction: heat current (flow of thermal energy) is driven by temperature gradients



[Consider an infinitesimal slab whose faces are at different temperatures.
→ Heat flows across slab.

Define heat current $I_h \equiv \frac{\text{thermal energy flow}}{\text{time}}$

Difference of temperature causes heat flow:

$$I_h = \frac{(\text{temp. difference})(\text{area of slab})}{(\text{thickness})}\left(\begin{array}{c}\text{material}\\ \text{dependent}\\ \text{constant}\end{array}\right)$$

[argument: if double thickness, then ΔT is halved for each half]
[also: insulation!]

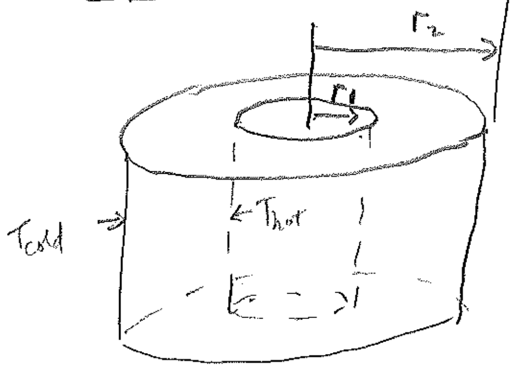
$$I_h = -\frac{\Delta T \cdot A \cdot k}{\Delta x}$$

$k =$ thermal conductivity $\left(\frac{J}{s \cdot m \cdot K}\right)$ $\begin{cases} \text{silver} \approx 400 \\ \text{glass} \approx 1 \\ \text{air} \approx 0.023 \end{cases}$

Define heat current density $J_h = \frac{I_h}{A}$ [per unit area]

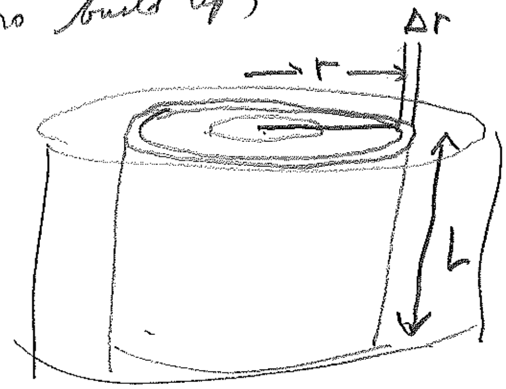
$$J_h = -\frac{k\,\Delta T}{\Delta x} \xrightarrow[\Delta x \to 0]{} -k\frac{dT}{dx} \quad \text{(Fourier's law)}$$

Application: insulation around a hot pipe



Assume steady-state (indep. of time)

Energy conservation
$\Rightarrow$ heat flow in = heat flow out
(no build up)



Heat flows radially $\Rightarrow$ choose infinitesimal slabs as thin cylindrical shells of radius r & thickness Δr.

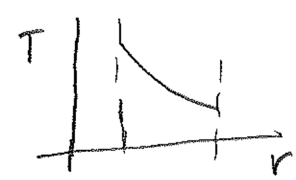
Area of shell $A = (2\pi r) L$

Heat flow through shell $I = -kA \frac{dT}{dr}$

$$= -2\pi k L \, r \frac{dT}{dr}$$

I must indep of t and indep of r, = constant

$$\frac{dT}{dr} = -\frac{I}{2\pi k L r} \qquad \text{[separable]}$$



$$T = -\frac{I}{2\pi k L} \log r + C_1$$

Apply boundary conditions.

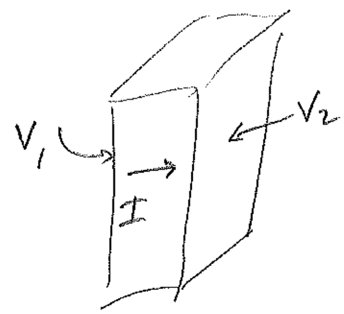
$$T_{hot} = -\frac{I}{2\pi k L} \log r_1 + C_1$$

$$T_{cold} = -\frac{I}{2\pi k L} \log r_2 + C_1$$

$$T_{hot} - T_{cold} = \frac{I}{2\pi k L}(\log r_2 - \log r_1) = \frac{I}{2\pi k L} \log\left(\frac{r_2}{r_1}\right)$$

$$\text{Heat flow } I = \frac{2\pi k L (T_{hot} - T_{cold})}{\log\left(\frac{r_2}{r_1}\right)}$$

3. Electrical conduction: electrical current (flow of charge) driven by voltage gradients



Consider an infinitesimal slab of resistive material whose faces are at different voltage

$\Rightarrow$ current flows through slab

Define electrical current $I = \dfrac{\text{charge transfer}}{\text{time}}$

Voltage difference causes current flow

$$I = -\frac{(\Delta V) A \sigma}{\Delta x}$$

σ = electrical conductivity

Current density $J = \dfrac{I}{A}$

x-component $J_x = -\sigma \dfrac{\partial V}{\partial x} = \sigma E_x$ since $E_x = -\dfrac{\partial V}{\partial x}$

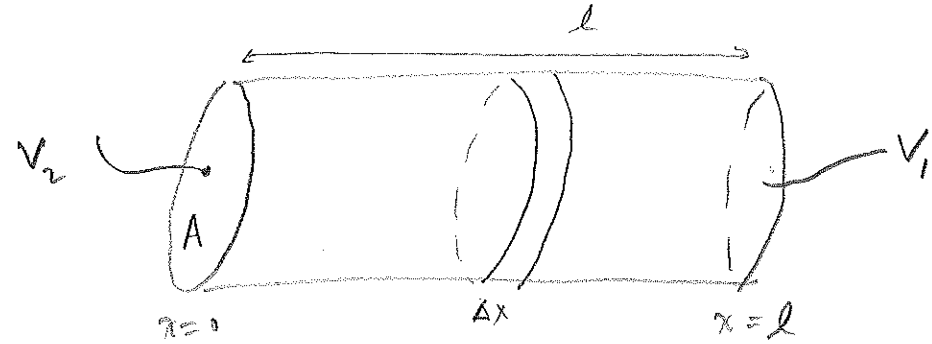
3 dimensions $\vec{J} = \sigma \vec{E}$ (Ohm's law)

ρ = electrical resistivity $\equiv \dfrac{1}{\sigma}$

$$\vec{E} = \rho \vec{J}$$

Cu: $\rho = 1.7 \times 10^{-18}$ Ωm
Ge $\rho = 0.46$
glass $\rho = 10^{10}$

Application: cylindrical resistor



Assume steady state.

$$I = -\sigma A \frac{dV}{dx}$$

I is independent of x, otherwise charge would build up

$$\frac{dV}{dx} = -\frac{I}{\sigma A}$$

$$V = -\frac{I}{\sigma A} x + C_1$$

Boundary conditions: $V_2 = V(0) = C_1$

$$V_1 = V(l) = -\frac{I}{\sigma A} \quad C_1$$

$$V_2 - V_1 = I\left(\frac{l}{\sigma A}\right)$$

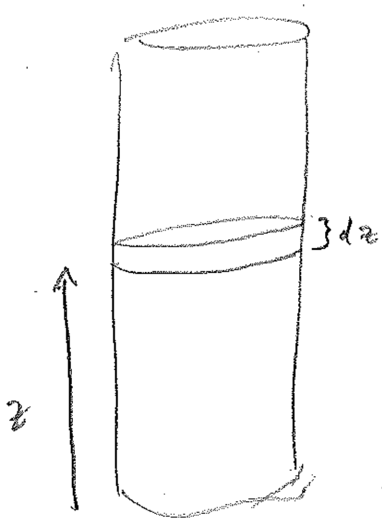
but $\Delta V = IR$ so

resistance of cylinder is $R = \dfrac{l}{\sigma A} = \dfrac{\rho l}{A}$

4. Continuum mechanics

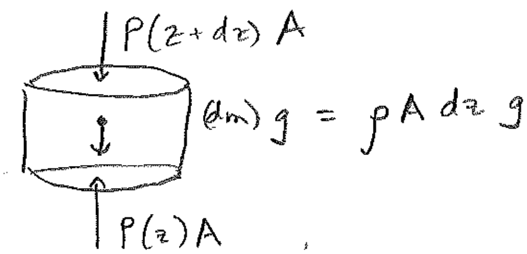
strategy:
- break system into infinitesimal elements
- draw free body diagram for each element
- apply $\vec{F} = m\vec{a}$ to each element

Static column of fluid:



P = Pressure = force/area, ρ = mass/volume

FBD:



Static ⇒ each element in mechanical equilibrium

$$F^z_{net} = P(z)A - \underbrace{P(z+dz)}_{P(z) + \frac{dP}{dz}dz + \dots} A - \underbrace{(dm)}_{\rho A\, dz}\, g = 0$$

$$\Rightarrow \quad \frac{dP}{dz} = -\rho g \quad \text{(hydrostatic eqn.)}$$

To proceed, need to know how P and ρ are related (equation of state)

(a) Assume incompressible fluid.

$\rho = \text{const}$, independent of P and $\therefore z$

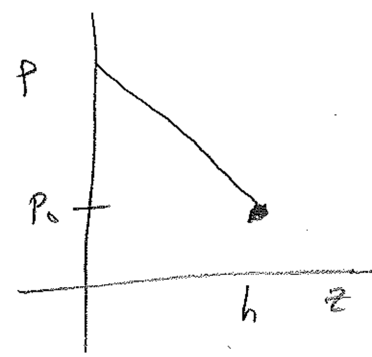
$$\frac{dP}{dz} = -\rho g$$

$$P = -\rho g z + C_1$$

Boundary condition: $P = P_0$ at top of column of fluid $z = h$

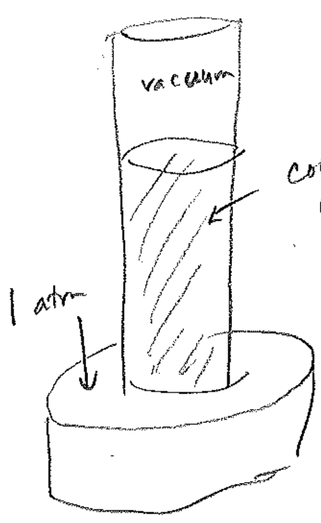
$$P_0 = -\rho g h + C_1$$

$$\Rightarrow \quad P = P_0 + \rho g (h - z)$$



Let's compute atmospheric pressure.

Atmosphere supports 760 mm of mercury



$$\underbrace{P(0)}_{1\text{ atm}} = \underbrace{P_0}_{0} + \underbrace{\rho g h}_{(13.5\ \frac{g}{cm^3})(9.8\ \frac{m}{s^2})(0.76\,m)}$$

$$\underbrace{(13.5\ \frac{g}{cm^3})(9.8\ \frac{m}{s^2})(0.76\,m)}_{1.013 \times 10^5\ Pa}$$

$\left(1 \text{ Pascal} = \frac{N}{m}\right)$

[pressure increases by 1 atmosphere every 10 m of water]

$$\rho g = (1\ \tfrac{g}{cm^3})(9.8\ \tfrac{m}{s^2}) = 10\ \frac{N}{m^3} = \frac{0.1 \text{ atm}}{10\,m}$$

If fluid is not incompressible, ρ depends on P
(equation of state)

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(b) ideal gas eqn of state $PV = NRT$

$$\begin{cases} N = \#\text{ moles} \\ R = N_A k_B = 8.31 \frac{J}{K \cdot mole} \end{cases}$$

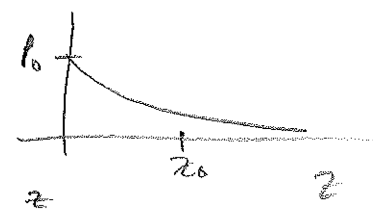
mass density $\rho = \frac{mN}{V}$, m = mass per mole

$$\Rightarrow \quad P = \rho \frac{RT}{m}$$

$\therefore$ hydrostatic eqn becomes $\frac{dP}{dz} = -\left(\frac{mg}{RT}\right) P$

Assume isothermal atmosphere $\Rightarrow$ T = const $\Rightarrow$ 1 dependent variable

$$P(z) = A e^{-\frac{mgz}{RT}}$$



Apply b.c. $P = P_0$ at $z = 0$

$$P(z) = P_0 e^{-\frac{mgz}{RT}} = P_0 e^{-\frac{z}{z_0}} \quad \text{(Barometric eqn)}$$

$$z_0 = \frac{RT}{mg} \approx \frac{(8.31 \frac{J}{K \cdot mo})(300\,K)}{(.028\, kg/mole)(9.8\, m/s^2)} \approx 9100\,m \approx 9\,km$$

N_2: 14×2

fall off slower for lighter species (H, He)