# β-decay

Tritium is (barely) unstable to $\beta^-$ decay

$$^3H \longrightarrow {}^3He + e^- + \bar{\nu}$$

$$Q = \Delta(^3H) - \Delta(^3He) = 14.950 - 14.931 = 0.019 \text{ MeV}$$

$$\left[\begin{array}{rl} m(^3_1H) = & 2809.4496 \text{ MeV} \\ m_e + m(^3_2He) = & \underline{2809.4310} \\ & 0.0186 \end{array}\right]$$

[ Interesting: ³He has 2 protons

repulsion $\frac{Ke^2}{r} \sim \frac{200 \text{ MeV}}{137 \cdot r} \Big|_{r=1.8 \text{ fm}} \sim 0.8 \text{ MeV}$

but $m_n > m_p + m_e$ by 0.8 MeV

Near perfect cancellation ]

very little phase space $\Rightarrow$ $T_{\frac{1}{2}} = 12$ years.

[Initially, it was thought that]

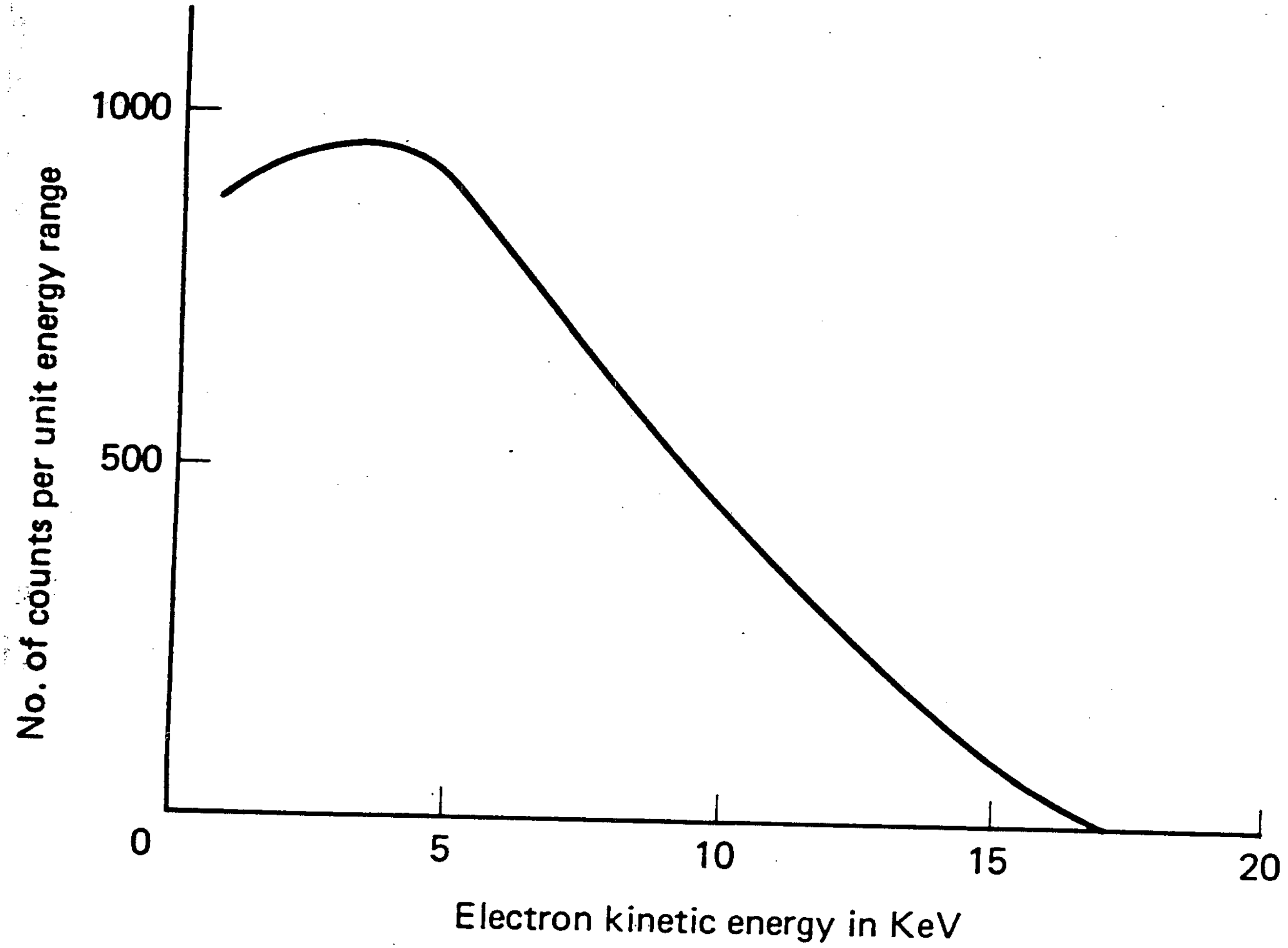
$$^3H \rightarrow {}^3He + e^-$$

2 particle decay $\Rightarrow$ kinetic energies fixed by $E, \vec{p}$ cons.

$m_e \ll m(^3He) \Rightarrow \quad K_{e^-} \simeq Q = 18.6$ keV

Experiments show a "continuous β-spectrum" [Chadwick 1914]

$$0 < K_{e^-} < 18.6 \text{ keV}$$



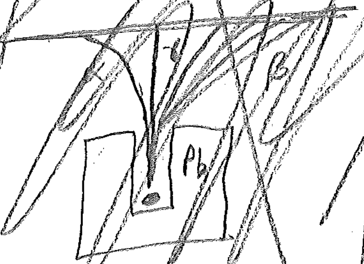
The beta decay spectrum of tritium (${}^{3}_{1}\mathrm{H} \rightarrow {}^{3}_{2}\mathrm{He}$). (*Source:* G. M. Lewis,

Fig. 1.6 Griffiths

~~Problem:~~

~~Continuous β-decay spectrum~~

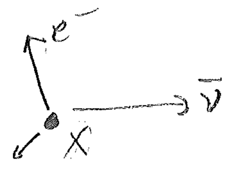
~~[disc. Chadwick 1914]~~ ~~[cf [illegible] fn [illegible]]~~



Bohr: energy non conservation?

Pauli (1930) postulated existence of a neutral massless particle
"neutrino"
which carries of some of the energy

$${}^{A}_{Z}X \to {}^{A}_{Z+1}Y\ e^-\ \bar{\nu}$$



[NB, not $K_\nu$]

$$Q \equiv \Delta(X) - \Delta(Y) = K_Y + K_e + E_{\bar\nu}$$

[NB don't include $m_\nu$ in Q]

$e^-$ and $\nu$ carry off most of kinetic energy (if momenta are comparable)

$\left(0 = \vec{p}_Y + \vec{p}_e + \vec{p}_\nu\right.$ and $\left.K_Y = \frac{p_Y^2}{2m_Y} \ll K_e, E_\nu\right)$

$$\Rightarrow K_e + E_\nu \approx Q$$

$$0 \leq K_e \leq Q - m_\nu c^2$$

[For 3He, $Q = 18.6$ keV. Since $(K_e)_{max} \gtrsim 18 \Rightarrow m_\nu \lesssim 500$ eV
Segre, Nuclei & Particles, 334]

$\nu$ is also needed to conserve angular momentum!
n, p, $e^-$ all spin $\frac{1}{2}$ (but neutron not known in 1930!)
$\Rightarrow \nu$ has $J = 1/2$

At any rate the earliest reference I know to the new particle is Heisenberg's mention of 'your neutrons' in a letter to Pauli[97] dated 1 December. More details are found in Pauli's letter (its main part follows) 1930 of 4 December to a gathering of experts on radioactivity in Tübingen.[60]

> Dear radioactive ladies and gentlemen,
> I have come upon a desperate way out regarding the 'wrong' statistics of the N- and the Li 6-nuclei, as well as to the continuous $\beta$-spectrum, in order to save the 'alternation law' of statistics* and the energy law. To wit, the possibility that there could exist in the nucleus electrically neutral particles, which I shall call neutrons, which have spin 1/2 and satisfy the exclusion principle and which are further distinct from light-quanta in that they do not move with light velocity. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any case not larger than 0.01 times the proton mass.—The continuous $\beta$-spectrum would then become understandable from the assumption that in $\beta$-decay a neutron is emitted along with the electron, in such a way that the sum of the energies of the neutron and the electron is constant.
>
> There is the further question, which forces act on the neutron? On wave mechanical grounds . . . the most probable model for the neutron seems to me to be that the neutron at rest is a magnetic dipole with a certain moment $\mu$. Experiments seem to demand that the ionizing action of such a neutron cannot be bigger than that of a $\gamma$-ray, and so $\mu$ may not be larger than $e \times 10^{-13}$ cm.
>
> For the time being I dare not publish anything about this idea and address myself confidentially first to you, dear radioactive ones, with the question how it would be with the experimental proof of such a neutron, if it were to have a penetrating power equal to or about ten times larger than a $\gamma$-ray.
>
> I admit that my way out may not seem very probable *a priori* since one would probably have seen the neutrons a long time ago if they exist. But only he who dares wins, and the seriousness of the situation concerning the continuous $\beta$-spectrum is illuminated by my honored predecessor, Mr. Debye, who recently said to me in Brussels: 'Oh, it is best not to think about this at all, as with new taxes'. One must therefore discuss seriously every road to salvation.—Thus, dear radioactive ones, examine and judge.—Unfortunately I cannot appear personally in Tübingen since a ball** which takes place in Zürich the night of the sixth to the seventh of December makes my presence here indispensable . . . Your most humble servant, W. Pauli'.

Recall: for an unstable particle, width $\Gamma = \frac{\hbar}{\tau}$

$$\underline{\frac{\text{Decay prob.}}{\text{time}}} \equiv \lambda = \frac{1}{\tau} = \frac{\Gamma}{\hbar} = \frac{1}{\hbar} \sum_{\substack{\text{all final}\\ \text{states}}} d\Gamma$$ ← partial width

Fermi's golden rule $d\Gamma = 2\pi \rho \, |\text{amplitude}|^2$

$\rho$ = density of final states

Fermi theory of weak interactions (~1932): $X \to Y e^- \bar{\nu}_e$

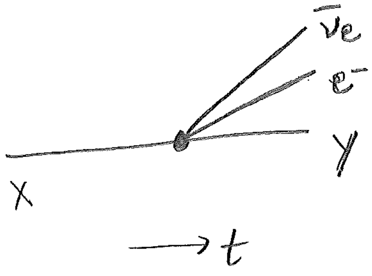
$\Rightarrow$ amplitude $= G_F \mathcal{M}$

Fermi constant $G_F = \underbrace{1.1664 \times 10^{-5}}_{\text{"weak"}} \text{GeV}^{-2} \cdot (\hbar c)^3$

$\left[ \frac{1}{M_W^2} \sim \frac{1.5E-4}{(\text{GeV})^2} \right]$

matrix element $\mathcal{M} = \langle Y e^- \bar{\nu}_e | O | X \rangle$ ↩ initial state

↗ final state ↑ some operator



$n \to p \bar{e} \bar{\nu}_e$ (880s)

$\mu^- \to e^- \bar{\nu}_e \nu_\mu$ (2.2E-6s)

$\Lambda \to p \pi^-$ (2.6E-10s)

$\tau^- \to \nu_\tau \mu^- \bar{\nu}_\mu$ (2.9E-13s)

Difference in $\tau$ due to phase space

## β-decay phase space

Must sum (integrate) over all possible momenta of final states $Y, e, \bar{\nu}_e$ subject to conservation of momentum & energy.

- Assume initial particle at rest $\Rightarrow \quad 0 = \vec{p}_Y + \vec{p}_e + \vec{p}_\nu$

  [i.e., given $\vec{p}_e + \vec{p}_\nu$, $\vec{p}_Y$ is fixed.]

- Define $Q = (m_X - m_Y - m_e)c^2$

  Cons of energy $\Rightarrow \quad Q = K_e + K_Y + E_\nu$

  (includes rest energy of ν, if any)

- Since $m_Y \gg m_e \Rightarrow K_Y \approx 0 \quad \Rightarrow K_e + E_\nu - Q = 0$

- Put universe in a (fictitious) box of volume $V$

  $\Rightarrow$ momenta are quantized.

$$\text{Density of momenta} = \frac{V}{h^3}(4\pi p^2 dp) = \frac{V}{(2\pi\hbar)^3} 4\pi p^2 dp$$

$$= \frac{V}{2\pi^2\hbar^3} p^2 dp$$

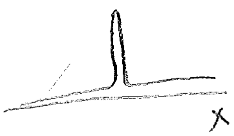
Phase space density

$$\rho = \left(\frac{V}{2\pi^2\hbar^3} p_e^2 dp_e\right)\left(\frac{V}{2\pi^2\hbar^3} p_\nu^2 dp_\nu\right) \underbrace{\delta(K_e + E_\nu - Q)}$$

Dirac delta function enforces energy conservation

1) $\delta(x) = 0$ unless $X = 0$

2) $\int \delta(x)\,dx = 1$



$$\rho(E)\,dE = \left(\prod \frac{V}{(2\pi\hbar)^3} d^3\vec{p}\right)$$

$$\Rightarrow \rho(E) = \rho(E)\,\Delta E\, f(\,) = \left(\prod \frac{V}{(2\pi\hbar)^3} d^3\vec{p}\right)\delta(\,)$$

$$\Gamma = \int d\Gamma = 2\pi \int \rho\, G_F^2 |M|^2$$

$$= \frac{V^2 G_F^2}{2\pi^3\hbar^6} \int p_e^2\, dp_e\, p_\nu^2\, dp_\nu\, \delta(K_e + E_\nu - Q)\, |M|^2$$

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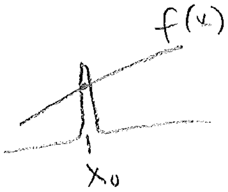
Assume neutrinos are massless $\Rightarrow p_\nu = \frac{E_\nu}{c}$

$$\int p_\nu^2\, dp_\nu\, \delta(K_e + E_\nu - Q)$$

$$= \frac{1}{c^3}\int E_\nu^2\, dE_\nu\, \delta(K_e + E_\nu - Q)$$

For any $f(x)$, $\int f(x)\,\delta(x - x_0) = f(x_0)$



$\therefore$

$$= \frac{1}{c^3}(Q - K_e)^2$$

$$\Rightarrow \Gamma = \frac{V^2 G_F^2}{2\pi^3\hbar^6 c^3} \int p_e^2\, dp_e\, (Q - K_e)^2 |M|^2$$

$$\mathcal{M} = \langle Y\, e^- \bar{\nu}_e | O | X \rangle$$

Both $e^-$ + $\bar{\nu}_e$ described by plane waves $\psi \sim \frac{1}{\sqrt{V}} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$

$\frac{1}{\sqrt{V}}$ is present so that $\int |\psi|^2 d^3x = 1$

Hence $|\mathcal{M}|^2 \sim \frac{\#}{V^2}$ which cancels $V^2$ in numerator

[Segre 9-5.4, p. 350]

$\Rightarrow$ set $V = 1$

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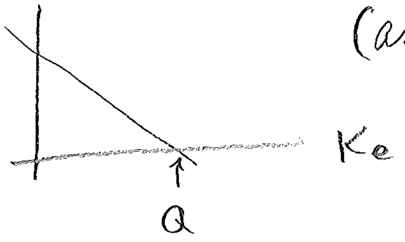
$$d\Gamma = \frac{G_F^2 |\mathcal{M}|^2}{2\pi^3 \hbar^6 c^3} (Q - K_e)^2 p_e^2 \, dp_e$$

$\frac{d\Gamma}{dp_e}$ = (prob of decay to an electron within range $p_e$ to $p_e + dp_e$) $\sim |\mathcal{M}|^2 (Q - K_e)^2 p_e^2$

$$\frac{\sqrt{\frac{d\Gamma}{dp_e}}}{p_e} \sim (Q - K_e)|\mathcal{M}|$$ ← theoretical prediction

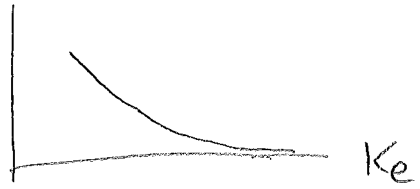
Kurie plot: $\frac{\sqrt{\frac{d\Gamma}{dp_e}}}{p_e}$ vs $K_e$

Expect



(assuming $|\mathcal{M}|$ is fairly insensitive to $p_e$)

Finite resolution of detector



If neutrino has mass then [HW]

correction factor for finite $m_\nu$

$$\frac{\sqrt{\frac{d\Gamma}{dp_e}}}{p_e} \sim |M|(Q-K_e)\left[\underbrace{1-\frac{(m_\nu c^2)^2}{(Q-K_e)^2}}_{\text{vanishes if } Q-K_e=m_\nu}\right]^{\frac{1}{4}}$$

$$\Rightarrow (K_e)_{max} = Q - m_\nu$$

→ see ~~plot~~ fig 6.2 ⓐ perfect + ⓑ finite resolution

$$\underline{\underline{m_\nu < 0.3\,eV}}$$

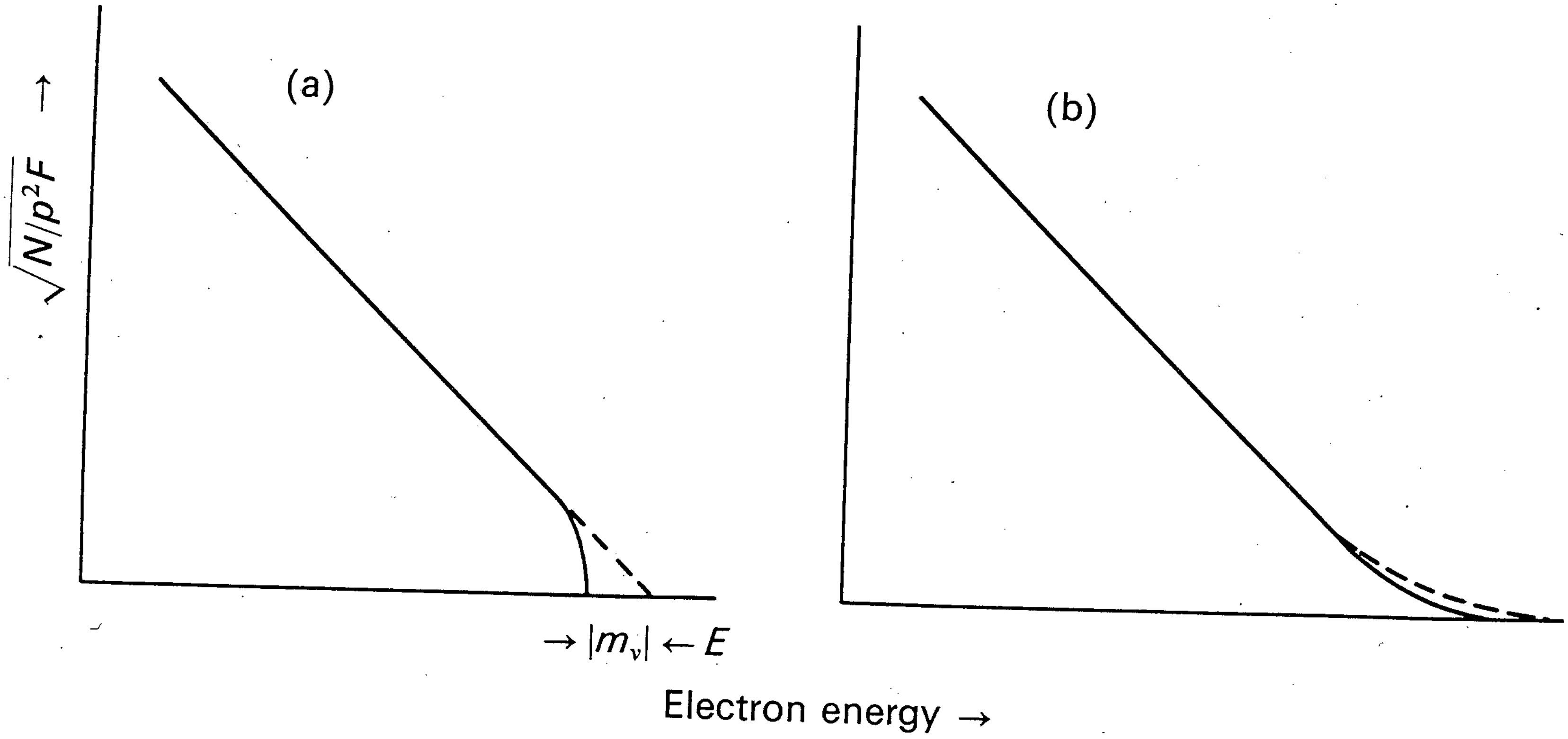


**Fig. 6.2** Kurie plot of allowed transition for zero and finite neutrino mass: (a) pe
resolution, (b) finite resolution.

Perkins

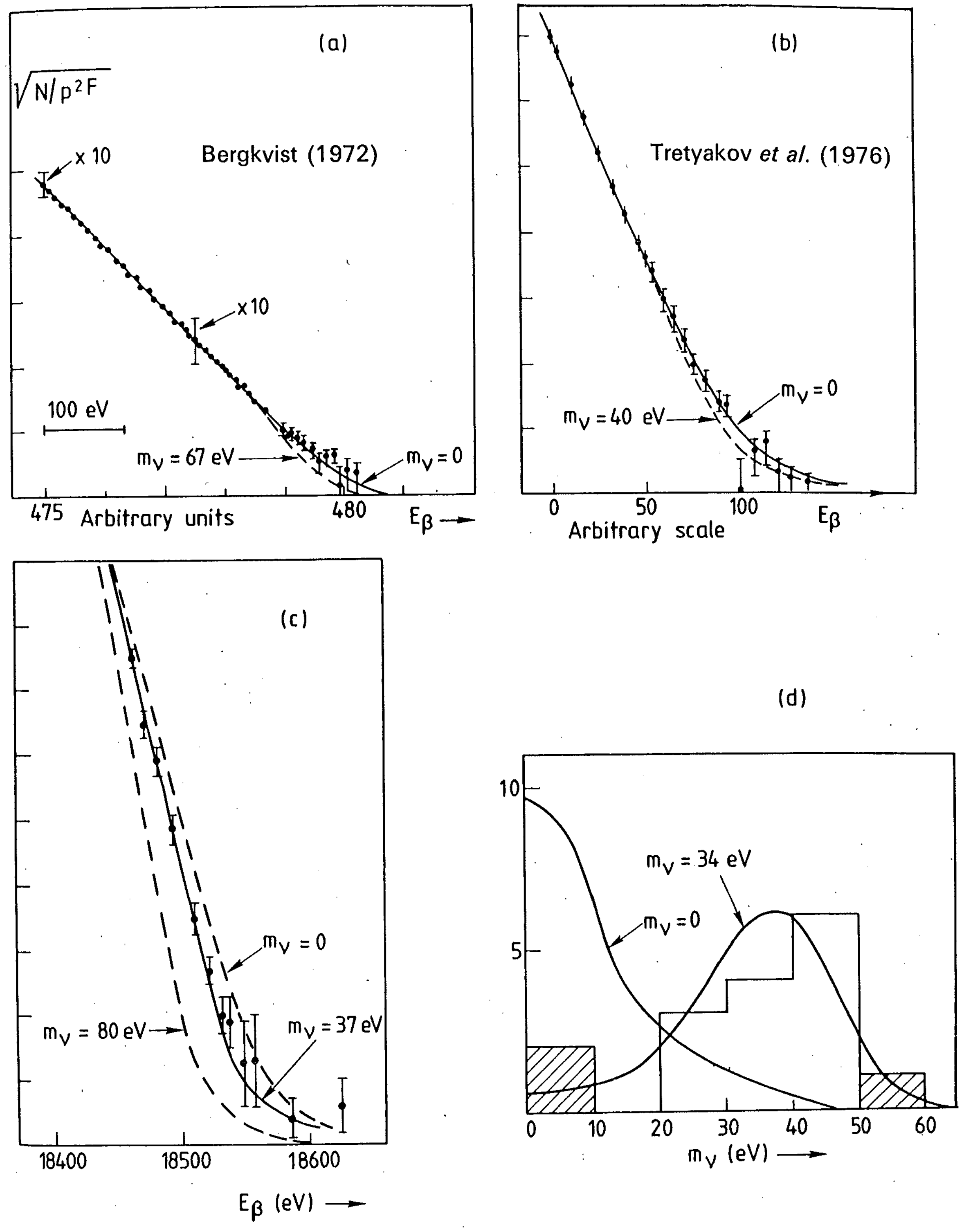


**Fig. 6.3** Kurie plots from recent measurements of tritium $\beta$-decay. (a) Bergkvist (1972); (b) Tretyakov *et al.* (1976); (c) and (d) Lyubimov *et al.* (1980). (c) shows the average of several runs, and (d) the "best" mass estimate for $m_\nu$ from each of 16 runs. The expected distributions for $m_\nu = 0$ and $m_\nu = 35$ eV/$c^2$ are indicated.

Perkins

$$\Gamma = \int d\Gamma = \frac{G_F^2}{2\pi^3 \hbar^6 c^3} \int_0^{(p_e)_{max}} dp_e \, p_e^2 \, (Q - K_e)^2 \, |M|^2$$

↖ depends on $\vec{p}_e, \vec{p}_\nu$ subject to $E_\nu = Q - K_e$

[HW: calc $(p_e)_{max}$]

To simplify, assume $|M|^2$ is relatively insensitive to momenta.

Also assume that $Q \gg m_e c^2$ so, electron is relativistic ($E_e = p_e c$)

Show that [HW]

$$\Gamma = (\text{const})\, G |M|^2 Q^5 \qquad [\text{Sargent's rule}]$$

[↑ high power explains huge difference between τ decay & μ decay

don't go into detail]

$\mu^- \Rightarrow \tau = 2.2E-6s$

$\tau^- \Rightarrow \tau = 2.9E-13s$

Ratio = 7.6 E6

$$\left[\Gamma \simeq \frac{G_F^2 |M|^2 Q^5}{60\pi^3 (\hbar c)^6}\right]$$

$$16.8 = \frac{m_\tau}{m_\mu} = \left(\frac{1776}{105.66}\right)^5 = 1.3E6$$

(Factor of 5.6 discrepancy)

[exact relativistic calc gives

$$\frac{1}{\tau_\mu} = \Gamma_\mu = \frac{G^2 (m_\mu c^2)^5}{192\pi^3 (\hbar c)^6}$$ ]

Problem later

## Using relativistic field theory approach

Srednicki

$$\langle f|i\rangle = (2\pi)^4\,\delta^4(\Sigma k)\; iT$$

$$\langle f|i\rangle^2 = VT\;(2\pi)^4\,\delta^4(\Sigma k)\,|T|^2$$

$$\langle i|i\rangle = 2mV$$

$$\langle f|f\rangle = 2E_iV$$

$$\text{Prob} = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} = \frac{T(2\pi)^4\,\delta^4(\Sigma k)\,|T|^2}{2m \quad \prod 2E_iV}$$

$$d\Gamma = \frac{\text{prob density}}{\text{time}} = \frac{(2\pi)^4\,\delta^4(\Sigma k)\,|T|^2}{2m}\prod\frac{\frac{V}{(2\pi)^3}\int d^3k_i}{2E_iV}$$

$$= \frac{(2\pi)^4\,\delta^4(\Sigma k)\,|T|^2}{2m}\left(\prod_i\int d\tilde{k}_i\right) \qquad d\tilde{k}_i = \frac{d^3k_i}{(2\pi)^3(2E_i)}$$

---

$$X \rightarrow Ye^-\bar{\nu}_e$$ Y gets momentum but no energy $\Rightarrow E_Y = m_Y$

Griffiths Golden Rule for decays

$$d\Gamma = \frac{(2\pi)\;|T|^2}{(2m_X)(2m_Y)}\underbrace{\int\frac{d^3k_e}{(2\pi)^3\,2E_e}\int\frac{d^3k_\nu}{(2\pi)^3\,2E_\nu}}\;\delta(K_e+E_\nu-Q)$$

$$\frac{k_e^2\,dk_e}{2\pi^2(2E_e)} \qquad \frac{k_\nu^2\,dk_\nu}{2\pi^2(2E_\nu)}$$

This essentially same as non-relativistic calculation (Fermi Golden rule) except for factors of $(2m_X)(2m_Y)(2E_e)(2E_\nu)$. But the relativistic $|T|^2 \approx (\underbrace{2m_X\,2m_Y}_{\text{eq }(12.19)}\;\underbrace{2E_e2E_\nu}_{\text{eq }(12.21)})$(angular dependence) so these cancel out.

[illegible]