Nuclear models

[observe that more abundant nuclei generally have smallest mass excess → greater binding energy ⇒ more stable]

$$\text{neutral atom}\quad E_{rest} = Z m_H c^2 + N m_n c^2 + \underbrace{\begin{pmatrix}\text{internal}\\ \text{kinetic}\\ \text{energy}\end{pmatrix}_{>0} + \begin{pmatrix}\text{attractive}\\ \text{potential}\\ \text{energy}\end{pmatrix}_{<0}}$$

total must be negative, else nucleus would be unstable to decay into p + n

$= -B$

$$\text{binding energy}\quad B = Z m_H c^2 + N m_n c^2 - E_{rest}$$

$$= Z(938.8 \text{ MeV}) + N(939.6 \text{ MeV}) - A(931.5 \text{ MeV}) - \Delta$$

$$= Z(7.29 \text{ MeV}) + N(8.07 \text{ MeV}) - \Delta$$

$Z = \frac{A}{2} - \frac{1}{2}(A-2Z)$
$N = \frac{A}{2} + \frac{1}{2}(A-2Z)$

$$= A(7.68 \text{ MeV}) + (A-2Z)(0.39 \text{ MeV}) - \Delta$$

If $Z \approx N = \frac{1}{2}A$ and $\Delta \approx 0$, then

$$B \approx (7.7 \text{ MeV})\, A$$

ie binding energy proportional to A

$\frac{B}{A} \sim 8$ MeV [look at curve ⇒]

[Actually $\frac{B}{A}$ higher for middle nuclei, less at ends ⇒ Explain!]

"Curve of binding energy"

[use table of Δ to compute + plot $\frac{B}{A}$ for most abundant nuclei w/ $A < 32$]

$$\frac{E_{rest}}{Z m_H c^2 + N m_n c^2} = 1 - \frac{B}{Z m_H c^2 + N m_n c^2} \simeq 1 - \frac{B}{A(940 \text{ MeV})} \approx 1 - 0.01$$

Nuclei are ~1% less massive than sum of constituents

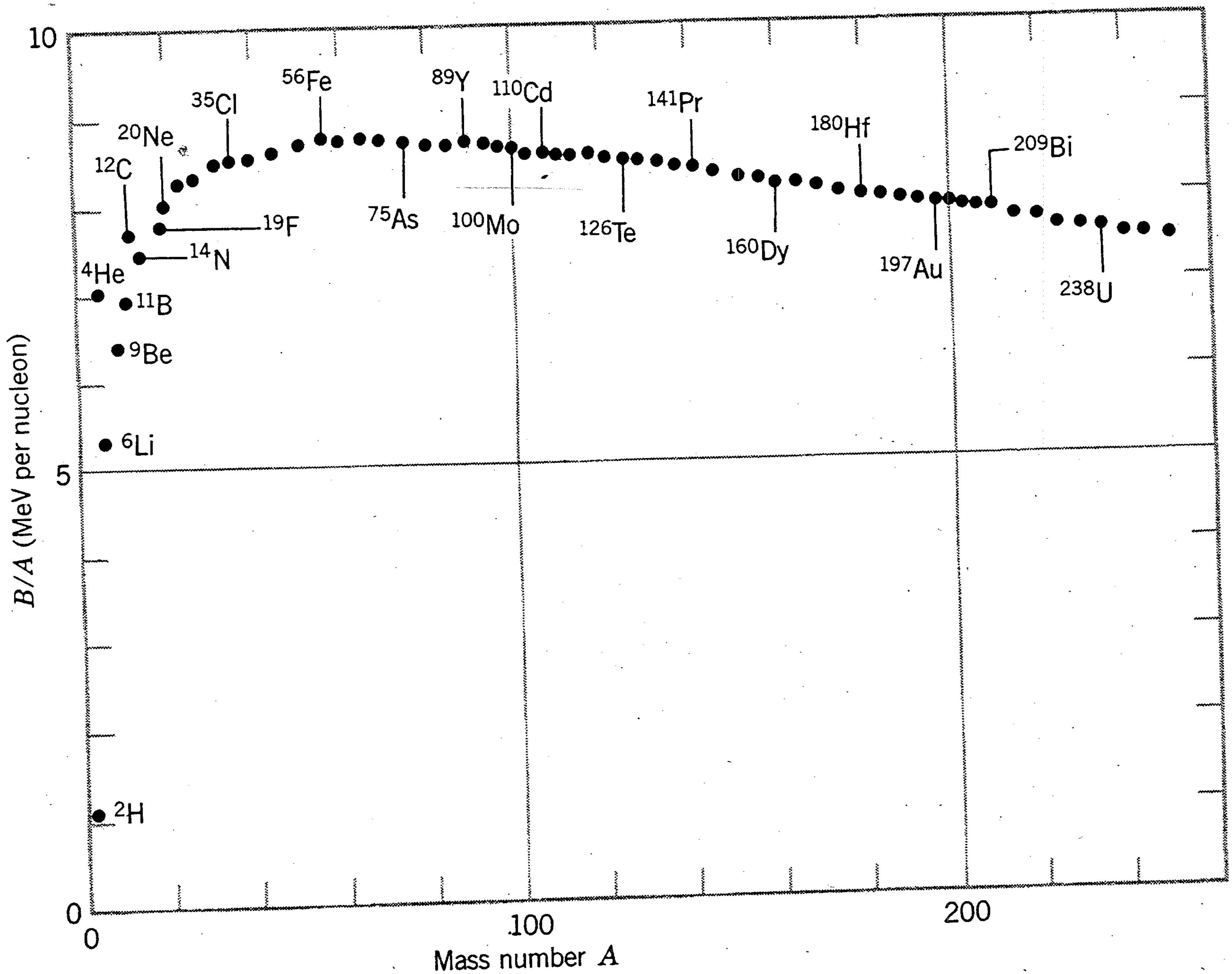
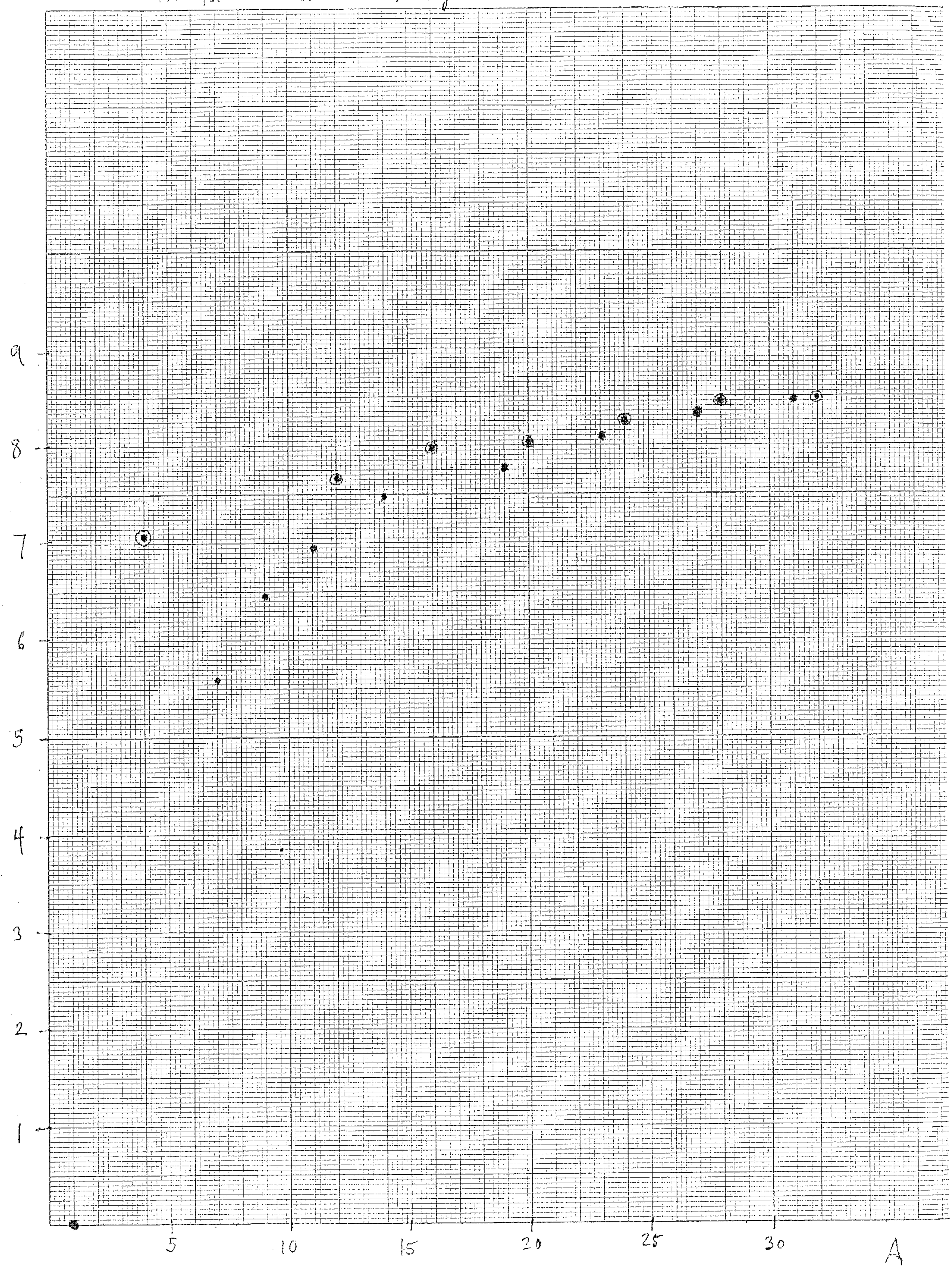


Figure 3.16 The binding energy per nucleon.

B/A for most abundant isotopes

⊙ = even-even nuclides



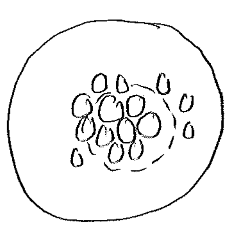
[Try to understand binding energy]

[Force primarily responsible for existence of nucleus is

<u>strong nuclear force</u>

• acts between pairs of nucleons (pp, pn, nn)

• short range) range << size of nucleus



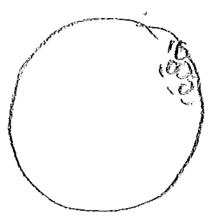
Suppose each nucleon acts on r others for some r

total # of pair ("bonds") is $\frac{1}{2} r A$

Potential energy $= -$(# pairs)(energy of "bond")

$$\Rightarrow B \approx c_1 A \qquad \text{(volume term)}$$

Then $\frac{B}{A} \approx c_1$ explains why binding energy per nucleon is $\approx$ const (due to short range nature of force, else $B \sim A^2$)



Nucleons on surface form fewer "bonds"

"missing bonds" $\sim$ # of nucleons on surface $\sim 4\pi R^2 \sim A^{2/3}$

$$B \approx c_1 A - c_2 A^{2/3} \quad \text{(surface term)}$$

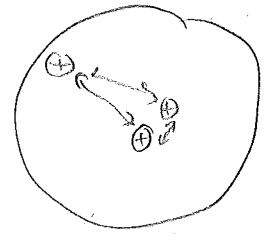
Then $\frac{B}{A} \approx c_1 - \frac{c_2}{A^{1/3}}$ explains why curve drops down for small A

Surface term causes nucleus to minimize surface area $\Rightarrow$ sphere

Liquid drop model of nucleus

Electromagnetic force

- acts between pairs of protons
- long range : affects all protons in nucleus



- repulsive, $\uparrow U$, reduces binding energy

Treat protons as uniformly-distributed charge $Q = Ze$

Potential energy of a sphere of charge $U = \frac{3}{5}\frac{KQ^2}{R} = \underbrace{\left(\frac{3}{5}\frac{Ke^2}{r_0}\right)}_{c_3}\frac{Z^2}{A^{1/3}}$

$$B \approx c_1 A - c_2 A^{2/3} - c_3 \frac{Z^2}{A^{1/3}}$$

$$\frac{B}{A} \approx c_1 - \frac{c_2}{A^{1/3}} - c_3 \frac{Z^2}{A^{1/3}}$$

explains why $\frac{B}{A}$ decreases for large Z, A & why nuclei eventually become unstable

[electromagnetic force eventually wins because it acts on all pairs of protons whereas strong only acts on neighboring nucleons]

Estimate $c_3 = \frac{3}{5}\frac{Ke^2}{r_0} = \frac{3}{5}\frac{(8.99E9\ J\cdot m/C^2)(1.6E-19\ C)^2}{(1.2E-15\ m)} =$

$= 1.15E-13\ J$ [eg Krane, p.67]

$= 0.72\ MeV$

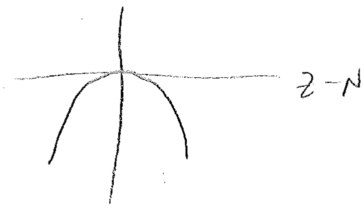
$$\left[\frac{3}{5}\frac{Ke^2}{\hbar c}\frac{\hbar c}{r_0} = \frac{3}{5}\alpha\frac{197\ MeV\,f}{1.2\ fm}\right]$$

~~EN~~ 9
NMY

<u>Symmetry term</u> (origins quantum mechanical)

acts to keep # protons & neutrons approx equal

$$\sim -\frac{(Z-N)^2}{A}$$



$$\sim -\frac{(A-2Z)^2}{A}$$

<u>Semi-empirical binding energy formula</u> (Bethe-Weizsäcker)
(fits data well for $A \gtrsim 15$)

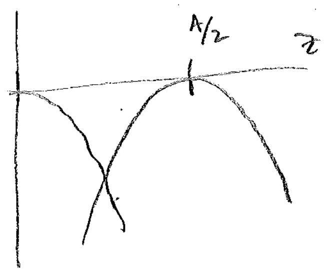
$$B = \underbrace{c_1 A - c_2 A^{2/3}}_{\text{strong}} \underbrace{- c_3 \frac{Z^2}{A^{1/3}}}_{\text{electromagnetic}} \underbrace{- c_4 \frac{(A-2Z)^2}{A} + \delta_{pair}}_{\text{quantum effects}}$$

[Krane]
$c_1 \approx 15.5$ MeV
$c_2 \approx 16.8$ MeV
$c_3 = 0.72$ MeV
$c_4 = 23$ MeV
$c_5 = 34$ MeV

$$\delta_{pair} = \begin{cases} c_5 A^{-3/4} & Z, N \text{ even} \\ 0 & Z \text{ or } N \text{ odd} \\ -c_5 A^{-3/4} & Z, N \text{ odd} \end{cases}$$

3rd & 4th terms determine the line of stability $\Rightarrow$

$$B = c_1 A - c_2 A^{2/3} - \underbrace{c_3 \frac{Z^2}{A^{1/3}}}_{\text{favors } Z \ll N} - \underbrace{c_4 \frac{(A-2Z)^2}{A}}_{\text{favors } Z = \frac{A}{2}} + \delta_{pair}$$



favors $Z \ll N$

more effective at large A

$\Downarrow$

heavy nuclei have fewer protons than neutrons

favors $Z = \frac{A}{2}$

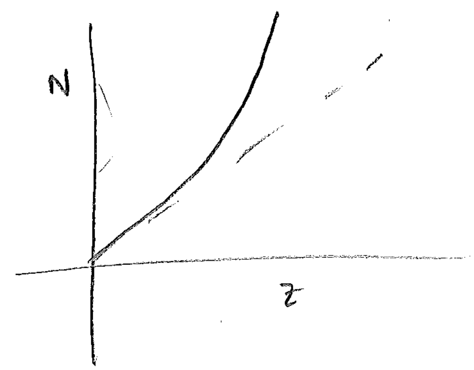
or $Z = N$

more effective at small A

$\Downarrow$

light nuclei have $Z \approx N$

^4He, ^{12}C, ^{16}O

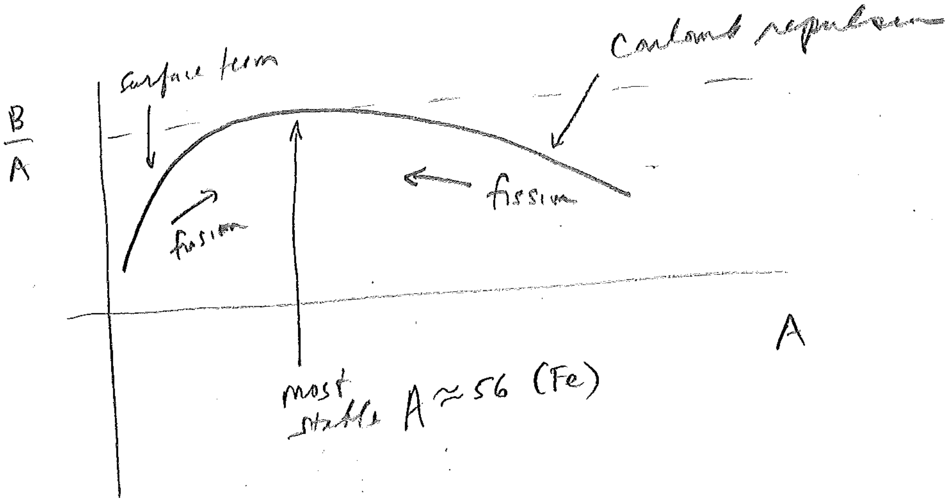


Binding energy per nucleon

$$\frac{B}{A} = c_1 - c_2 A^{-1/3} - c_3 \frac{Z^2}{A^{4/3}} - c_4 \frac{(A-2Z)^2}{A^2} + \frac{\delta_{pair}}{A}$$

assume $Z \approx N \approx \frac{A}{2}$

$$\frac{B}{A} = c_1 - \frac{c_2}{A^{1/3}} - c_3 A^{2/3}$$



$X \rightarrow Y + Z$ iff $m_X > m_Y + m_Z$

A nucleus is stable only if its mass is less than the sum of masses of nuclei into which it could decay

[First 2 terms of semi-empirical explained by liquid drop.
3rd term → Coulomb repulsion.

4th term → symmetry term }
5th term → pairing term } QM effects.

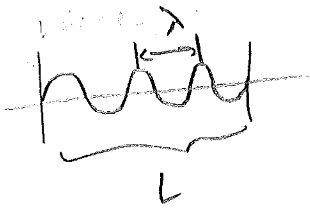
How understand symmetry term?]

Fermi gas model of nucleus

treat protons + neutrons as non-interacting fermions confined to V.

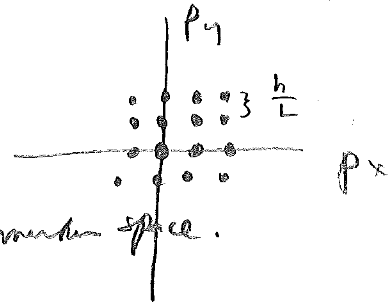
For simplicity, cube of side $L = V^{1/3}$

p + n have de Broglie wavelength $\lambda = \frac{h}{p}$ where $\lambda = \frac{L}{n}$, $n \in \mathbb{Z}$



(periodic b.c.) $p = \frac{h}{\lambda} = \frac{hn}{L}$

3 dimensions $\vec{p} = \frac{h}{L}(n_x, n_y, n_z)$



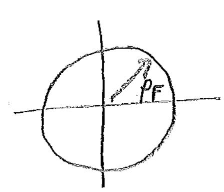
Each state takes up volume $\left(\frac{h}{L}\right)^3$ in momentum space.

Identical fermions occupy different states, filling out a sphere of radius p_F

Total # of identical (spin-$\frac{1}{2}$) fermions $N = 2 \left(\frac{L}{h}\right)^3 \int dp_x\, dp_y\, dp_z$

$$= 2\,\frac{V}{h^3}\int_0^{p_F} 4\pi p^2\, dp = \frac{8\pi V p_F^2}{3h^3} = \frac{V p_F^2}{3\pi^2 \hbar^3}$$

$$\Rightarrow p_F = \left(\frac{3\pi^2 N \hbar^3}{V}\right)^{1/3} = \left(\frac{9\pi}{4}\,\frac{N}{A}\right)^{1/3}\frac{\hbar}{r_0}$$



Recall $V = \left(\frac{4}{3}\pi r_0^3\right)A$

Is the Fermi gas relativistic or nonrelativistic?

$$E = \sqrt{(cp_F)^2 + (mc^2)^2}$$

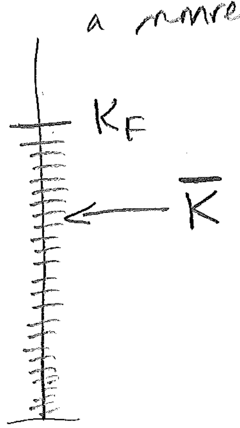
$cp_F \ll mc^2$ non rel
$cp_F \gg mc^2$ ultra rel

Estimate $cp_F \sim \frac{c\hbar}{r_0} = \frac{(197\text{ MeV fm})}{(1.2\text{ fm})} \sim 170\text{ MeV} \ll 940\text{ MeV}$

Treat it as nonrel (all momenta are $\leq p_F$)

$$K = \frac{p^2}{2m} \Rightarrow K_F = \frac{p_F^2}{2m} = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi N}{4A}\right)^{2/3}$$

Show that the average kinetic energy of a nonrelativistic fermi gas is $\bar{K} = \frac{3}{5}K_F$ [HW]



Therefore total kinetic energy of N identical (spin-½) fermions is

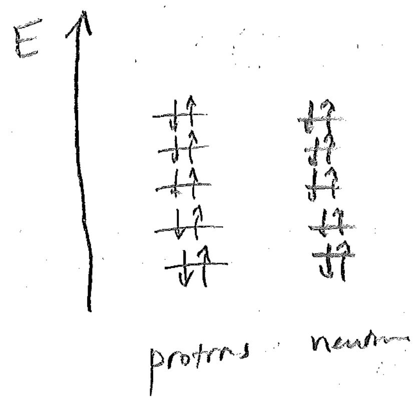
$$K_{tot} = \left(\frac{3}{5}K_F\right)N = \frac{3\hbar^2}{10mr_0^2}\left(\frac{9\pi}{4}\right)^{2/3}\left(\frac{N}{A}\right)^{\frac{5}{3}} \cdot A$$

$$\frac{\hbar^2}{mr_0^2} = \frac{(\hbar c)^2}{(mc^2)r_0^2} = \frac{(197\text{ MeV fm})^2}{(940\text{ MeV})(1.2\text{ fm})^2} \cong 29\text{ MeV}$$

Include protons and neutrons

$$K_{tot} = \frac{3}{10}\left(\frac{9\pi}{4}\right)^{2/3}\left[\left(\frac{Z}{A}\right)^{5/3} + \left(\frac{A-Z}{N}\right)^{5/3}\right](29\text{ MeV}) \cdot A$$

First, assume $Z = N = \frac{1}{2} A$

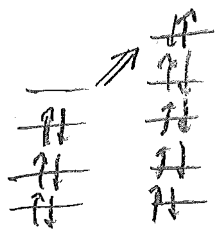


$$K_{tot} = \underbrace{\frac{3}{10}\left(\frac{9\pi}{4}\right)^{2/3}\left[\left(\tfrac{1}{2}\right)^{5/3} + \left(\tfrac{1}{2}\right)^{5/3}\right]}_{0.7} (29\,\mathrm{MeV})\,A$$

$$= (20\,\mathrm{MeV})\,A$$

This contributes to the volume term
implies that the potential energy per nucleon
is $\sim -35\,\mathrm{MeV}$

If $Z \neq N$, kinetic energy increases

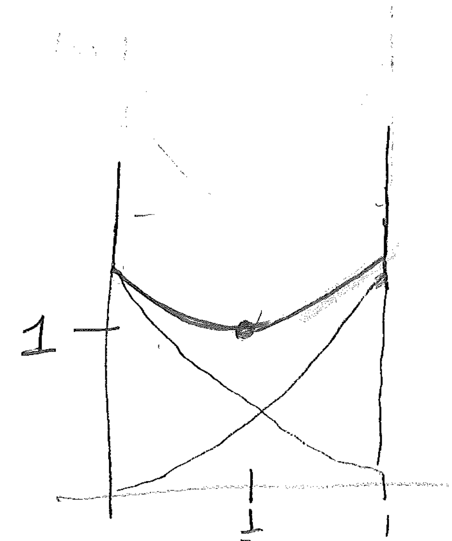


proton neutron

Estimate how much:

Let $x = \frac{Z}{A}$ then

$$K = (20\ \text{MeV})\, A \cdot \frac{f(x)}{f(\frac{1}{2})}$$



where $f(x) = x^{5/3} + (1-x)^{5/3}$

$$\frac{f(x)}{f(\frac{1}{2})} = 1 + \#\,(x-\tfrac{1}{2})^2 + \dots$$

[answer: $\# = \frac{20}{9}$]

[Calculate this & use it to estimate c_4 in semi-empirical;

(Hint: it doesn't agree too well but is right order of magnitude]

[$c_4 = 11$ MeV whereas actual $c_4 = 24$ MeV

Calc of asymmetry term in semi-empirical formula

Kinetic energy of nucleus

$$\vec{k} = \frac{2\pi}{L}\vec{n}$$

$$\frac{4\pi}{3} n_{max}^3 = \frac{N}{2}$$

$$\epsilon_F = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \left(\frac{3N}{8\pi}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$U = \frac{3}{5} N \epsilon_F$$

$r_0 = 1.2$ fm (Segrè)
$= 1.5$ fm (Fermi)

For a nucleus: $V = A V_0$, $V_0 = \frac{4}{3}\pi r_0^3$

$$\epsilon_F = \underbrace{\frac{\hbar^2 c^2}{2 m c^2 r_0^2}\left(\frac{9\pi}{4}\right)^{2/3}}_{C_0} \left(\frac{N}{A}\right)^{2/3}$$

C_0: 53 MeV (Segrè), 34 MeV (Fermi)

$\longrightarrow N_p = N_n = \frac{A}{2} \Rightarrow \epsilon_F \simeq 21$ MeV/fe

if only neutrons, $\epsilon_F = \begin{Bmatrix} 53 \\ 34 \end{Bmatrix}$

(27 MeV say K+K, $r_0 = 1.3$ f)

$$U_{Nuc} = \frac{3}{5} N_p C_0 \left(\frac{N_p}{A}\right)^{2/3} + \frac{3}{5} N_n C_0 \left(\frac{N_n}{A}\right)^{2/3}$$

$$= \frac{3}{5}\frac{C_0}{A^{2/3}}\Big[\underbrace{Z^{5/3} + (A-Z)^{5/3}}_{f(Z,A)}\Big]$$

$$\rho = Z - \frac{A}{2}$$

$$f(Z,A) = \left(\frac{A}{2}+\rho\right)^{5/3} + \left(\frac{A}{2}-\rho\right)^{5/3}$$

$$= \frac{A^{5/3}}{2^{3/2}}\left[1 + \frac{20}{9}\frac{\rho^2}{A^2}\right] = (.63)A^{5/3} + (.35)A^{-1/3}(A-2Z)^2$$

$$U_{Nuc} = \underbrace{(.38)C_0}_{\substack{20 \text{ MeV (Segrè)} \\ 13 \text{ MeV (Fermi)}}} A + \underbrace{(.21)C_0}_{\substack{11 \text{ MeV (Segrè)} \\ 7 \text{ MeV (Fermi)}}}\frac{(A-2Z)^2}{A}$$

If all neutrons, $(A = N_n, Z = 0)$ $U_{Nuc} = \frac{3}{5}C_0 A = \begin{Bmatrix} 32 \text{ MeV} \\ 20 \text{ MeV} \end{Bmatrix} A$,