Spin of a composite particle

Let 2 particles have spins $\vec{J}_1$ and $\vec{J}_2$

Total angular momentum of the system is $\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{L}$
where $\vec{L}$ = relative orbital angular momentum

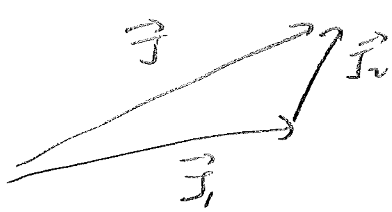
We'll assume the system is in the ground state, so $\vec{L} = 0$

Then the spin of the composite is $\vec{J} = \vec{J}_1 + \vec{J}_2$

"Addition of angular momentum"

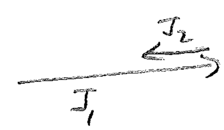
Classical approach

Assume $J_1 \geq J_2$



The magnitude of total spin J can range from a

minimum of $J_1 - J_2$



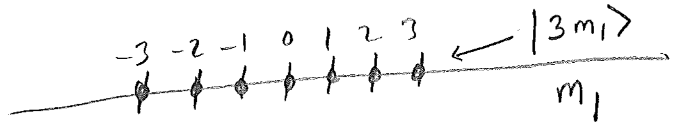
to a maximum of $J_1 + J_2$

Quantum

1^{st} particle has $J_1 = j_1 \hbar$ and is described by a multiplet $2j_1+1$ of states $|j_1, m_1\rangle$ where $J_{1z} = m_1 \hbar$ and m_1 ranges from $-j_1$ to j_1

Example

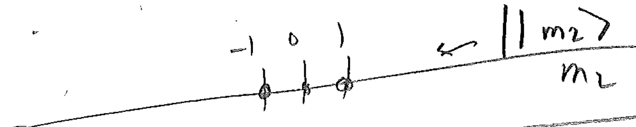
spin 3 or $\underset{\sim}{7}$



2^{nd} particle has $J_2 = j_2 \hbar$, described by $2j_2+1$ of states $|j_2\, m_2\rangle$ w/ $J_{2z} = m_2 \hbar$

Example

spin 1 or $\underset{\sim}{3}$



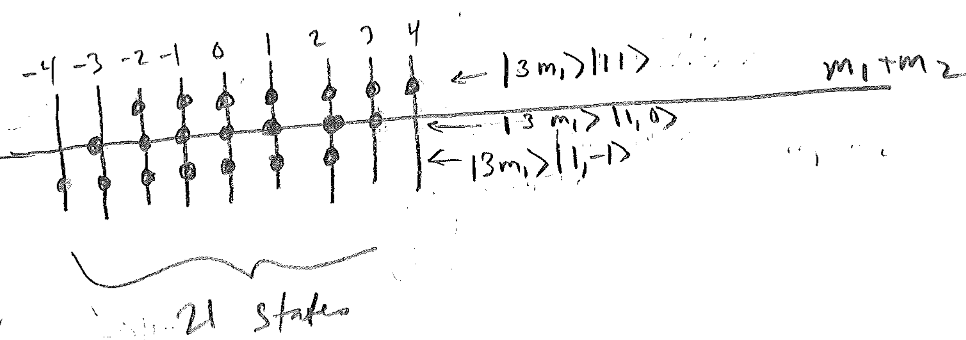
The two particles have a total number of states $= (2j_1+1)(2j_2+1)$ labelled by $|j_1\, m_1\rangle |j_2\, m_2\rangle$ $\begin{cases} m_1 = -j_1 \ldots j_1 \\ m_2 = -j_2 \ldots j_2 \end{cases}$

The states live in a product space $(\text{spin } j_1) \otimes (\text{spin } j_2)$ or $\underset{\sim}{(2j_1+1)} \otimes \underset{\sim}{(2j_2+1)}$

Let's plot all possible values of $m_1 + m_2$ for the product states

(spin 3) ⊗ (spin 1)

$\underset{\sim}{7} \otimes \underset{\sim}{3}$



The composite particle also has quantized spin

$$\left.\begin{array}{l} J = j\hbar \\ J_z = m\hbar \end{array}\right\}$$ and is described a multiplet $2j+1$ of states $|j, m\rangle$

What are the <u>possible</u> values of j?

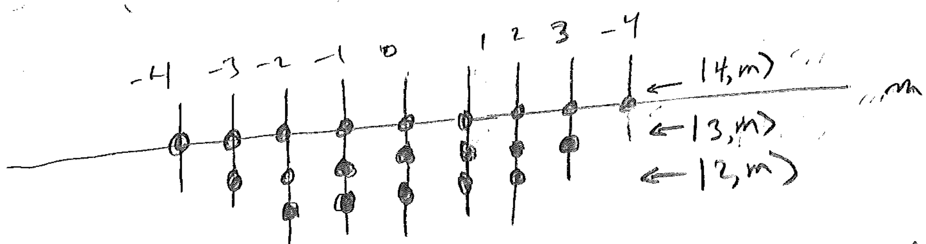
$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

$$J_z = J_{1z} + J_{2z} = m_1\hbar + m_2\hbar = (m_1 + m_2)\hbar$$

so $m = m_1 + m_2$ [this is why we plotted $m_1 + m_2$]

Since $m_{1,max} = j_1$ and $m_{2,max} = j_2$ $\Rightarrow$ $m_{max} = j_1 + j_2$

$\Rightarrow$ $j_{max} = j_1 + j_2$



21 states

spin 4 ⊕ spin 3 ⊕ spin 2

$\underset{\sim}{9} \oplus \underset{\sim}{7} \oplus \underline{5}$

In general

$(\text{spin } j_1) \otimes (\text{spin } j_2) = (\text{spin } j_1 + j_2) \oplus (\text{spin } j_1 + j_2 - 1) \oplus \ldots \oplus \text{spin } (j_1 - j_2)$

Decomposition of the product space in a sum of spaces [compare w/ classical]

<u>Conclusion</u> The composite particle can have any spin from $j_1 + j_2$ to $|j_1 - j_2|$.

[But subtlety: While $|4,4\rangle = |3,3\rangle|1,1\rangle$

$|4,3\rangle$ is neither $|32\rangle|11\rangle$ nor $|33\rangle|10\rangle$

but a linear combination as we'll now see]

Example

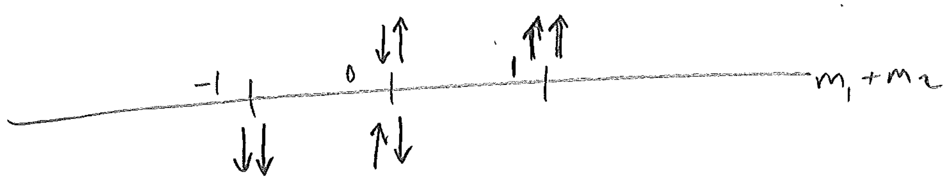
Product space of two (spin $\frac{1}{2}$) particles   e.g. $\underbrace{p \text{ and } n}_{\text{deuteron}}$ or $\underbrace{q \text{ and } \bar{q}}_{\text{meson}}$

$$(\text{Spin } \tfrac{1}{2}) \otimes (\text{Spin } \tfrac{1}{2}) \quad \text{or} \quad \underbrace{\underset{\sim}{2} \otimes \underset{\sim}{2}}_{4 \text{ states}}$$

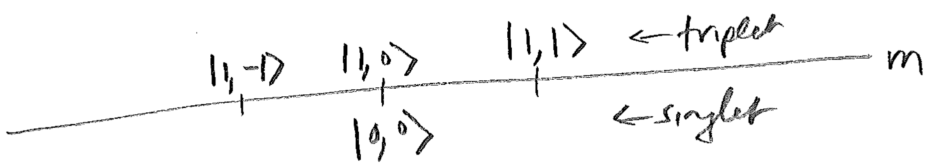
Denote $\uparrow = |\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$\downarrow = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

Then the four states of product space $|\tfrac{1}{2}, m_1\rangle |\tfrac{1}{2}, m_2\rangle$ are $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, and $\downarrow\downarrow$



But $\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}$



Match up states in the product space
with states in the space of total spin

$$|1,1\rangle = \uparrow\uparrow$$

Triplet ($j=1$) $\quad |1,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$

[Makes sense: we'll prove it in QM]

$$|1,-1\rangle = \downarrow\downarrow$$

Note: triplet state is symmetric under exchange of particles
[that's why we picked relative + sign]

Singlet ($j=0$) $\quad |0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

Singlet state is antisymmetric under exchange

$|1,0\rangle$ + $|0,0\rangle$ are entangled states:
[z-component of spin of neither particle is well-defined but they are correlated. "I don't know what my spin but whatever it is, its the opposite of yours". like enemies: "if you agree w/ it, I disagree".]

A composite particle made of 2 spin $\frac{1}{2}$ particles can have either $J=0$ or $J=1$

Example $\pi^+ = u\bar{d}$ has $J=0$ (scalar meson)
$\rho^+ = u\bar{d}$ has $J=1$ (vector meson)
deuteron ${}^2_1H = pn$ has $J=1$ [why? we'll see later]
(abundance 0.015%)

but note, deuteron has mag dipole moment of $0.857\mu_N$
($\approx \mu_p + \mu_n$
2.79
-1.91
0.88)

SU(2), S_2
$(\square\square, \square\square) + (\boxminus, \boxminus)$

If the 2 particles are identical, the entire wavefunction must be antisymmetric under exchange.

If both in same spatial state, then spin state must be antisymmetric ie $\uparrow\downarrow - \downarrow\uparrow$

Hence $J = 0$. for the system

(e.g. electronic ground state of helium)

[Colloquially we say spins must be antiparallel but more precisely we mean $|0,0\rangle$ not $|1,0\rangle$.

NB. excited states of helium can have $J=0$ (parahelium) or $J=1$ (orthohelium)]

Write decomposition in a more cumbersome manner

$$\begin{cases} |1\,1\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |1\,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |1\,-1\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \end{cases}$$

$$|0\,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

Called Clebsch-Gordan decomposition

$$\underbrace{|j\,m\rangle}_{\text{state of total spin}} = \sum_{\substack{j_1, j_2 \\ m_1, m_2 \\ (m_1+m_2=m)}} \underbrace{C^{j\,j_1\,j_2}_{m\,m_1\,m_2}}_{\text{C-G coefficient}} \underbrace{|j_1\,m_1\rangle}_{\text{state of 1st spin}} \underbrace{|j_2\,m_2\rangle}_{\text{state of 2nd spin}}$$

eg $C^{1\,\frac{1}{2}\,\frac{1}{2}}_{0,\,\frac{1}{2},\,-\frac{1}{2}} = \frac{1}{\sqrt{2}}$

[see Clebsch Gordan tables in PDB]

$$\begin{array}{c|c} & j \\ & m \\ \hline m_1 \quad m_2 & C \end{array}$$

Also $|j_1 m_1\rangle|j_2 m_2\rangle = \displaystyle\sum_{j=|j_1-j_2|}^{j_1+j_2} C^{j\,j_1\,j_2}_{m\,m_1\,m_2}\,|j\,m\rangle$

$1/2 \times 1/2$

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

$1 \times 1/2$

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2	Coefficients		
m_1 m_2			
⋮ ⋮			

1/2×1/2

m_1 m_2	J=1, M=+1
+1/2 +1/2	1

m_1 m_2	J=1, M=0	J=0, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=1, M=−1
−1/2 −1/2	1

1×1/2

m_1 m_2	J=3/2, M=+3/2
+1 +1/2	1

m_1 m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	J=3/2, M=−3/2
−1 −1/2	1

2×1/2

m_1 m_2	J=5/2, M=+5/2
+2 +1/2	1

m_1 m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	J=5/2, M=−5/2
−2 −1/2	1

3/2×1/2

m_1 m_2	J=2, M=+2
+3/2 +1/2	1

m_1 m_2	J=2, M=+1	J=1, M=+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	J=2, M=0	J=1, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=2, M=−1	J=1, M=−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	J=2, M=−2
−3/2 −1/2	1

2×1

m_1 m_2	J=3, M=+3
+2 +1	1

m_1 m_2	J=3, M=+2	J=2, M=+2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	J=3, M=−2	J=2, M=−2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	J=3, M=−3
−2 −1	1

1×1

m_1 m_2	J=2, M=+2
+1 +1	1

m_1 m_2	J=2, M=+1	J=1, M=+1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	J=2, M=−1	J=1, M=−1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	J=2, M=−2
−1 −1	1

3/2×1

m_1 m_2	J=5/2, M=+5/2
+3/2 +1	1

m_1 m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	J=5/2, M=−5/2
−3/2 −1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1 m_2	J=7/2, M=+7/2
+2 +3/2	1

m_1 m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	J=7/2, M=−7/2
−2 −3/2	1

3/2×3/2

m_1 m_2	J=3, M=+3
+3/2 +3/2	1

m_1 m_2	J=3, M=+2	J=2, M=+2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	J=3, M=−2	J=2, M=−2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	J=3, M=−3
−3/2 −3/2	1

2×2

m_1 m_2	J=4, M=+4
+2 +2	1

m_1 m_2	J=4, M=+3	J=3, M=+3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	J=4, M=−3	J=3, M=−3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	J=4, M=−4
−2 −2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

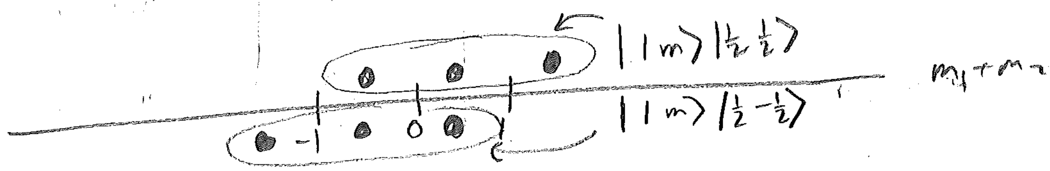
$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

Example; product space (spin 1) $\otimes$ (spin $\frac{1}{2}$) = (spin $\frac{3}{2}$) $\oplus$ (spin $\frac{1}{2}$)

$$\underset{\sim}{3} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2}$$



Read From CG table:

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = |1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$\alpha^2+\beta^2=1$
normalized ✓

$$|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = |1,-1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

[in QM, students derive these coeffs]

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

[Given st of ~~sp~~ total spin $\frac{3}{2}$ & $S_z=\frac{1}{2}$, what is prob ~~it~~ spin st will have $S_z=0$? $|\sqrt{\tfrac{2}{3}}|^2=\tfrac{2}{3}$]

{ref: $\square \times \square \times \square = (\overset{s=3/2\ \text{sym}}{\square\square\square, \square\square\square}) + (\overset{s=\frac{1}{2}}{\square\square, \square\square})$ dim = 2 }

AA '9

Product space of 3 spin-$\frac{1}{2}$ particles

$$(\text{spin } \tfrac{1}{2}) \otimes (\text{spin } \tfrac{1}{2}) \otimes (\text{spin } \tfrac{1}{2}) = (\overset{S}{\text{spin } 1} \oplus \overset{A}{\text{spin } 0}) \otimes \text{spin } \tfrac{1}{2}$$

$$\underbrace{\underset{\sim}{2} \times \underset{\sim}{2} \times \underset{\sim}{2}}_{8 \text{ states}} \qquad = (\overset{S}{\text{spin } 1 \otimes \text{spin } \tfrac{1}{2}}) + (\overset{A}{\text{spin } 0 \otimes \text{spin } \tfrac{1}{2}})$$

$$= (\text{spin } \tfrac{3}{2}) \oplus (\text{spin } \tfrac{1}{2})_S \oplus (\text{spin } \tfrac{1}{2})_A$$

eg $\uparrow\downarrow\uparrow$ etc.

$$= \underset{\sim}{4} + \underset{\sim}{2}_S + \underset{\sim}{2}_A$$

---

spin-$\frac{3}{2}$ states

$$|\tfrac{3}{2}, \tfrac{3}{2}\rangle = |1, 1\rangle |\tfrac{1}{2} \tfrac{1}{2}\rangle = (\uparrow\uparrow)\uparrow = \uparrow\uparrow\uparrow$$

$$|\tfrac{3}{2} \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}} |1, 1\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}} |1, 0\rangle |\tfrac{1}{2} \tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{1}{3}} (\uparrow\uparrow)\downarrow + \sqrt{\tfrac{2}{3}} \sqrt{\tfrac{1}{2}} (\uparrow\downarrow + \downarrow\uparrow)\uparrow = \frac{1}{\sqrt{3}} (\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$

$$|\tfrac{3}{2} -\tfrac{1}{2}\rangle = \ldots = \frac{1}{\sqrt{3}} (\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow)$$

$$|\tfrac{3}{2}, -\tfrac{3}{2}\rangle = \downarrow\downarrow\downarrow$$

completely symmetric under exchange of particles

$(\text{spin-}\frac{1}{2})_S$ state

normalised

$$|\tfrac{1}{2} \tfrac{1}{2}\rangle_S = \sqrt{\tfrac{2}{3}} |1 1\rangle |\tfrac{1}{2} -\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}} |1 0\rangle |\tfrac{1}{2} \tfrac{1}{2}\rangle =$$

$$= \sqrt{\tfrac{2}{3}} \uparrow\uparrow\downarrow - \sqrt{\tfrac{1}{6}} (\uparrow\downarrow + \downarrow\uparrow)\uparrow = \frac{1}{\sqrt{6}} (2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle_S = \sqrt{\tfrac{1}{3}} |1, 0\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}} |1, -1\rangle |\tfrac{1}{2} \tfrac{1}{2}\rangle = \frac{1}{\sqrt{6}} (\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2 \downarrow\downarrow\uparrow)$$

symm'c in 1st + 2nd

$(\text{spin-}\frac{1}{2})_A$ state

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle_A = |0, 0\rangle |\tfrac{1}{2} \tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow)\uparrow = \frac{1}{\sqrt{2}} (\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$|\tfrac{1}{2} -\tfrac{1}{2}\rangle_A = |0, 0\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}} (\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

antisymm'c in 1st + 2nd

N.B. None of the states is completely antisymmetric, ie under exchange of all 3 particles

$\Rightarrow$ can't put 3 identical fermions in same spatial state (Pauli)

Ground state of $^3He = ppn$ (0.000138 % abundance)

must be antisymmetric in the 2 protons ie

<u>must</u> be in $(\text{spin}\,\tfrac{1}{2})_A$ states

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle_A = \frac{1}{\sqrt{2}}\left(p^\uparrow p^\downarrow - p^\downarrow p^\uparrow\right) n^\uparrow$$

$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle_A = \frac{1}{\sqrt{2}}\left(p^\uparrow p^\downarrow - p^\downarrow p^\uparrow\right) n_\downarrow$$

[could have been $J = \frac{3}{2}$ but proton spins must be antiparallel]

Also magnetic dipole moment of helium $\sim$

$-2.13\,\mu_N$

(Segrè) ($\simeq \mu_n = -1.91\,\mu_N$)

tritium $\sim 2.98\,\mu_N$

$\sim \mu_p = 2.79\,\mu_N$

PHYSICS 280 *Problem Set 3* Due: Monday, February 11, 2013

1. Suppose you had two spin-2 particles, each in a state with $J_z = 0$. Assume that the orbital angular momentum of the system is zero. If you measured the total angular momentum of this system, what possible values could you get, and what is the probability of obtaining each value. [Check that they add up to 1.]

2. Suppose you had a spin-$\frac{3}{2}$ particle and a spin-2 particle. Assume that the total spin of the composite system is $\frac{5}{2}$ and its z-component is $-\frac{1}{2}$. Assume also that the orbital angular momentum of the system is zero. If you measured the z-component of the spin-2 particle, what possible values could you get, and what is the probability of obtaining each value? [Check that they add up to 1.]

3. (a) Consider four spin-$\frac{1}{2}$ particles. Write down the decomposition of the product space

$$\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = ?\oplus ?\oplus ?\oplus ?\oplus ?\oplus ?$$

What are the possible values of the total spin J, and how many times does each value occur?

(b) For each of the six representations found in part (a), write the expression for the $J_z = 0$ state $|j, 0\rangle$ in terms of a linear combination of direct product states (e.g. $\uparrow\downarrow\uparrow\uparrow$, etc.). You should use partial results derived in class for this. Your states $|j, 0\rangle$ should be correctly normalized. (They should also be orthogonal to one another.)

(c) Which of the representations above corresponds to the ground state of helium-4? Why?

4. Consider three (massive) spin-1 particles. What are the possible values of the total spin of the system (assuming no relative orbital angular momentum), and how many times does each value appear in the decomposition? Show that the total number of states in the decomposition is equal to the total number of product states.

For The New York Times

Science Meets Tabloid TV

Problem

$$2 \otimes 2 \otimes 2 \otimes 2 = (4 \oplus 2_S \oplus 2_A) \otimes 2$$
$$= (4 \otimes 2) + (2_S \otimes 2) + (2_A \otimes 2)$$
$$= 5 \oplus 3 \oplus 3_S \oplus 1_S \oplus 3_A \oplus 1_A$$

---

5 $$|2,0\rangle = \frac{1}{\sqrt{2}} |\tfrac{3}{2}, \tfrac{1}{2}\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}} |\tfrac{3}{2}, -\tfrac{1}{2}\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$
$$= \frac{1}{\sqrt{6}} (\uparrow\uparrow\downarrow\downarrow + \uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow\uparrow + \downarrow\downarrow\uparrow\uparrow)$$

---

3 $$|1,0\rangle = \frac{1}{\sqrt{2}} |\tfrac{3}{2}, \tfrac{1}{2}\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}} |\tfrac{3}{2}, -\tfrac{1}{2}\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$
$$= \frac{1}{\sqrt{6}} (\uparrow\uparrow\downarrow\downarrow + \uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow - \downarrow\downarrow\uparrow\uparrow)$$

3_S $$|1,0\rangle_S = \frac{1}{\sqrt{2}} \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle_S |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_S |\tfrac{1}{2}, \tfrac{1}{2}\rangle \right)$$
$$= \frac{1}{2\sqrt{3}} (2\uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow\uparrow - 2\downarrow\downarrow\uparrow\uparrow)$$

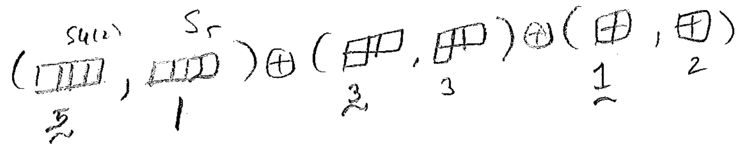
1_S $$|0,0\rangle_S = \frac{1}{\sqrt{2}} \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle_S |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_S |\tfrac{1}{2}, \tfrac{1}{2}\rangle \right)$$
$$= \frac{1}{2\sqrt{3}} (2\uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow + 2\downarrow\downarrow\uparrow\uparrow)$$

3_A $$|1,0\rangle_A = \frac{1}{\sqrt{2}} \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle_A |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_A |\tfrac{1}{2}, \tfrac{1}{2}\rangle \right)$$
$$= \frac{1}{4} (\uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow)$$

1_A: $$|0,0\rangle_A = \frac{1}{\sqrt{2}} \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle_A |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_A |\tfrac{1}{2}, \tfrac{1}{2}\rangle \right)$$
$$= \frac{1}{4} (\uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow\uparrow)$$
$$= \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \longleftarrow {}^4\mathrm{He}$$

NB: all are normalized and orthogonal

4 spin $\frac{1}{2}$ particles



$$2\otimes 2\otimes 2\otimes 2 = (4\oplus 2_S\oplus 2_A)\otimes 2$$
$$= (4\otimes 2)\oplus(2_S\otimes 2)\oplus(2_A\otimes 2)$$
$$= 5\oplus 3 \;\oplus\; 3_S\oplus 1_S \;\oplus\; 3_A\oplus 1_A$$

5: $|2,2\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = (\uparrow\uparrow\uparrow)\uparrow = \quad \uparrow\uparrow\uparrow\uparrow$

3: $|1,1\rangle = \frac{\sqrt{3}}{2}|\tfrac{3}{2}\tfrac{3}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \frac{1}{2}|\tfrac{3}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$

$= \frac{\sqrt{3}}{2}\uparrow\uparrow\uparrow\downarrow - \frac{1}{2\sqrt{3}}(\uparrow\uparrow\downarrow+\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow)\uparrow$

$= \frac{1}{2\sqrt{3}}(3\uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow - \uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow)$

3_S: $|1,1\rangle_S = |\tfrac{1}{2},\tfrac{1}{2}\rangle_S|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow\uparrow - \uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow)$

1_S: $|0,0\rangle_S = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle_S|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle_S|\tfrac{1}{2}\tfrac{1}{2}\rangle\right)$

$= \frac{1}{2\sqrt{3}}(2\uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow + 2\downarrow\downarrow\uparrow\uparrow)$

$\binom{4}{2}=6$

3_A: $|1,1\rangle_A = |\tfrac{1}{2}\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow)$

1_A: $|0,0\rangle_A = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2}-\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$

$= \frac{1}{2}(\uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow\uparrow)$

$= \underbrace{\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)}_{pp}\;\underbrace{\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)}_{nn} \leftarrow \boxed{\text{helium 4.}}$

SC-4

Let $\tilde{P}$ = cyclic perm operator

~~$\frac{1}{2}(2\uparrow\uparrow\downarrow\downarrow - \downarrow\uparrow\uparrow\downarrow) + \frac{3}{2}\downarrow\uparrow\uparrow\downarrow$~~

~~$\tilde{P}|1_A\rangle = \frac{1}{2}(\ldots)$~~ $\frac{1}{2}(-\uparrow\downarrow\uparrow\downarrow - 3\uparrow\downarrow\downarrow\downarrow)$

Neither $|1_A\rangle$ nor $|1_S\rangle$ are eigstates of $\tilde{P}$, but both are eigstates of $\tilde{P}^2$ w/ eigval 1, so can form eigstates of $\tilde{P}$ w/ eigval ± 1

Proof related to way spin chain works

Consider $|1\rangle \propto \sqrt{3}\,|1_S\rangle - 3\,|1_A\rangle$

$= \frac{1}{2}(2\uparrow\uparrow\downarrow\downarrow + 2\downarrow\uparrow\uparrow\downarrow + 2\downarrow\downarrow\uparrow\uparrow + 2\uparrow\downarrow\downarrow\uparrow$
$- 4\uparrow\downarrow\uparrow\downarrow - 4\downarrow\uparrow\downarrow\uparrow)$

3+

$= \uparrow\uparrow\downarrow\downarrow + \downarrow\uparrow\uparrow\downarrow + \downarrow\downarrow\uparrow\uparrow + \uparrow\downarrow\downarrow\uparrow - 2\uparrow\downarrow\uparrow\downarrow - 2\downarrow\uparrow\downarrow\uparrow$

normalized $|1\rangle = \frac{1}{2}|1_S\rangle - \frac{\sqrt{3}}{2}|1_A\rangle$

$= \frac{1}{\sqrt{12}}(\uparrow\uparrow\downarrow\downarrow + \downarrow\uparrow\uparrow\downarrow + \downarrow\downarrow\uparrow\uparrow + \uparrow\downarrow\downarrow\uparrow - 2\uparrow\downarrow\uparrow\downarrow - 2\downarrow\uparrow\downarrow\uparrow)$

also can explicitly see that $S_+|1\rangle = 0$

$\boxed{\tilde{P}|1\rangle = |1\rangle}$

orthog. comb

$|1\rangle' = \frac{\sqrt{3}}{2}|1_S\rangle + \frac{1}{2}|1_A\rangle$

$= \frac{1}{4}(2\uparrow\uparrow\downarrow\downarrow - 2\downarrow\uparrow\uparrow\downarrow + 2\downarrow\downarrow\uparrow\uparrow - 2\uparrow\downarrow\downarrow\uparrow$
$- \cancel{\uparrow\downarrow\uparrow\downarrow} - \cancel{\downarrow\uparrow\downarrow\uparrow})$

can also explicitly verify $S_+|1\rangle' = 0$

$= \frac{1}{2}(\uparrow\uparrow\downarrow\downarrow - \downarrow\uparrow\uparrow\downarrow + \downarrow\downarrow\uparrow\uparrow - \uparrow\downarrow\downarrow\uparrow)$

$\boxed{\tilde{P}|1\rangle' = -|1'\rangle}$