Kinematics

$1 \to 2$ $\quad X_0 \to X_1 X_2$ $\quad$ [class]
[HW, $\pi \to \mu\nu$]

$2 \to 1$ $\quad X_A X_p \to X_0$ (FT) $\quad$ [HW, $\pi p \to \Delta$]

$X_A X_B \to X_0$ (cm) $\quad$ [HW, $\bar{p}p \to \pi^0$]

$1 \to 3$ $\quad X_0 \to X_1 X_2 X_3$ $\quad$ [HW, $X \to Y e \nu$
$\mu \to e \nu \bar{\nu}$]

$2 \to 2$ $\quad X_A X_B \to X_1 X_2$

elastic, cm $\quad \alpha X \to \alpha X$ $\quad$ [class, Rutherford]
$\gamma e \to \gamma e$ $\quad$ [HW]
$\Big\}$ all p the same

elastic, FT $\quad \gamma e \to \gamma e$ $\quad$ [class]

inelastic, cm $\quad e^- e^+ \to \mu^- \mu^+$ $\quad$ [class, HW]
$e^- e^+ \to \gamma\gamma$ $\quad$ [exam?]
$\Big\}$ all E the same

inelastic, cm $\quad \nu p \to e^+ n$ $\quad$ [HW]

Four-momentum

4-vector $u^\mu = (u^0, u^1, u^2, u^3)$

Transforms as $u^\mu \to \Lambda^\mu{}_\nu u^\nu$ under Lorentz transformation

We'll often omit the index, simply writing u,
but it is implied by context

To distinguish 4-vectors from 3-vectors: u or u^μ vs $\vec{u}$ or u^i

---

[Primarily we'll be concerned w/]

4-momentum $p = \left(\frac{E}{c}, \vec{p}\right)$.

Usually set $c = 1$ as before (B-8): $p = (E, \vec{p})$

Conservation of 4-momentum of a system

perhaps later

$$P^{sys}_{init} = P^{sys}_{final}$$ (each component separately conserved)

$\Rightarrow \begin{cases} \text{conservation of energy} \\ \text{conservation of 3-momentum} \end{cases}$

Given two 4-vectors u and w

define Lorentz-invariant scalar product ("Lorentz scalar")

$$u \cdot w = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} u^\mu w^\nu$$

where Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ ← particle physics convention "mostly minus metric"

$$u \cdot w = u^0 w^0 - \vec{u} \cdot \vec{w}$$

$$u^2 \equiv u \cdot u = (u^0)^2 - \vec{u}^2 \qquad \vec{u}^2 = \vec{u} \cdot \vec{u} = |\vec{u}|^2$$

Use arrow to distinguish square of 3-vector vs 4-vector

[4-vectors and Lorentz scalars are a powerful tool.]

For a single particle:

$p = (E, \vec{p})$

$p^2 = E^2 - \vec{p}^{\,2}$

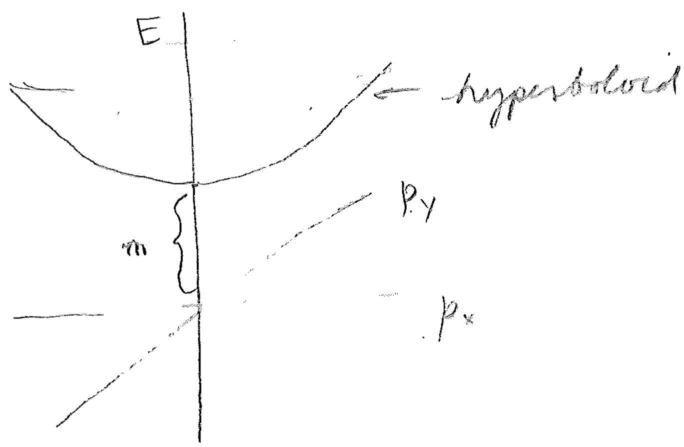
Recall from P2140: $E^2 = (c\vec{p})^2 + (mc^2)^2$ or $E^2 = |\vec{p}|^2 + m^2$

express E, p, m all in terms of energy, e.g. MeV

$\Rightarrow \boxed{p^2 = m^2}$ mass is Lorentz scalar

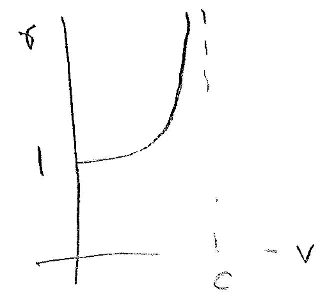
If a particle obeys $p^2 = m^2$ we say it is "on the mass shell" or "onshell"

$E^2 \; \vec{p}^{\,2} + m^2$



This is true for physical particles but not virtual ones

If we need to express $E, \vec{p}$ in terms of $\vec{v}$, use



$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$ gamma factor
time-dilation factor

$\vec{p} = \gamma m \vec{v}$ (relativistic momentum)

$E = \gamma m c^2$

$T = E - mc^2 = (\gamma - 1) mc^2$ (relativistic kinetic energy)

$$\frac{c\vec{p}}{E} \quad \frac{\vec{v}}{c}$$

omitting c's:

$$\frac{\vec{p}}{E} \quad \vec{v}$$

where $p, E \sim$ MeV and $v \sim$ dimensionless ($v = 1 \rightarrow$ speed of light)

[often not necessary or helpful to use these when applying energy & momentum conservation]

Non relativistic limits ($v \ll c$, $\gamma \approx 1$)

$$\gamma = 1 + \frac{1}{2}v^2 + \ldots$$

$$\vec{p} = \gamma m \vec{v} = m\vec{v} + \ldots$$

$$E = \gamma m = m + \frac{1}{2}m v^2 + \ldots$$

nonrelativistic kinetic energy

$$E = \sqrt{p^2 + m^2} = m\sqrt{1 + \frac{p^2}{m^2}} = m + \frac{p^2}{2m} + \ldots$$

Ultrarelativistic limit ($v \approx c$, $\gamma \gg 1$) [less familiar to you]

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

$$v = \sqrt{1 - \frac{1}{\gamma^2}} = 1 - \frac{1}{2\gamma^2} + \ldots$$

$$E = \sqrt{p^2 + m^2} = p\sqrt{1 + \frac{m^2}{p^2}} = p + \frac{m^2}{2p} + \ldots$$

[proton in LHC has $E = 7$ TeV, $\gamma = 7000$, $v = 1 - (10^{-8}) = 0.\underbrace{999\;999\;99}_{8\ 9\text{'s}}$]

Two-particle decay $\quad X_0 \to X_1 X_2$

Energy + mom. conservation completely determine kinematics

Consider rest frame of X_0

$$X_1 \longleftarrow \bullet \quad \bullet \longrightarrow X_2$$

$$\left.\begin{array}{lll} \text{mass shell condition:} & E_1^2 = m_1^2 + |\vec{p}_1|^2 \\ & E_2^2 = m_2^2 + |\vec{p}_2|^2 \\ \text{en. cons} & m_0 = E_1 + E_2 \\ \text{mom cons} & |\vec{p}_1| = |\vec{p}_2| \end{array}\right\} \begin{array}{l} \text{4 eqns,} \\ \text{4 unknowns} \end{array}$$

straightforward:
$$|\vec{p}_1|^2 = |\vec{p}_2|^2$$
$$E_1^2 - m_1^2 = E_2^2 - m_2^2$$
$$= (m_0 - E_1)^2 - m_2^2$$

$$\Rightarrow 2m_0 E_1 = m_0^2 - m_2^2 + m_1^2$$
$$E_1 = \frac{m_0^2 - m_2^2 + m_1^2}{2m_0}$$

4-momentum:
$$p_0 = p_1 + p_2$$
$$\Rightarrow p_0 - p_1 = p_2$$

"Square" $\quad p_0^2 - 2p_0 \cdot p_1 + p_1^2 = p_2^2$

$$m_0^2 - 2(\underbrace{E_0 E_1}_{m_0 E_1} - \underbrace{\vec{p}_0 \cdot \vec{p}_1}_{0}) + m_1^2 = m_2^2$$

$$\Rightarrow m_0^2 + m_1^2 - m_2^2 = 2m_0 E_1$$

By symmetry $\quad E_2 = \dfrac{m_0^2 - m_1^2 + m_2^2}{2m_0}$

check: $E_1 + E_2 = m_0$

$$T_1 = E_1 - m_1$$

$$= \frac{(m_0^2 - 2m_0 m_1 + m_1^2) - m_2^2}{2m_0}$$

$$= \frac{(m_0 - m_1)^2 - m_2^2}{2m_0}$$

$$= \left(\frac{m_0 - m_1 + m_2}{2m_0}\right)\underbrace{(m_0 - m_1 - m_2)}_{Q\text{ : total } T \text{ released}}$$

Fraction of Q going to m_1

$$\frac{T_1}{Q} = \frac{m_0 - m_1 + m_2}{2}$$

Frac'n to m_2

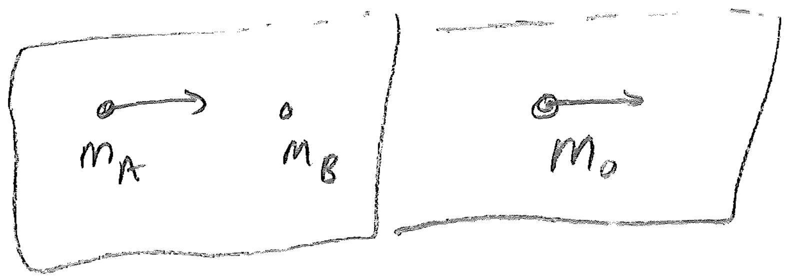
$$\frac{T_2}{Q} = \frac{m_0 - m_2 + m_1}{2}$$

If $m_1 \approx m_2$ then $\frac{T_1}{Q} = \frac{T_2}{Q} = \frac{1}{2}$

If $m_1 \approx m_0$ then $\frac{T_1}{Q} \approx 0, \ \frac{T_2}{Q} \approx 1$

[eg. α decay]

[HW]



Compute $\frac{T_A}{Q}$

$\pi p \to \Delta$ (Fermi)

β-decay

Initially it was thought that tritium decay was

[cf β-13]

$^3_1H \longrightarrow {}^3_2He + e^-$

[obeys charge + baryon # conservation]

kinetic energy released $Q = m_H - m_{He} - m_e$

$= \Delta(^3H) - \Delta(^3He) = 18.6$ keV

as we calculated earlier.

Initial state tritium at rest $\Rightarrow T_{He} + T_e = Q$

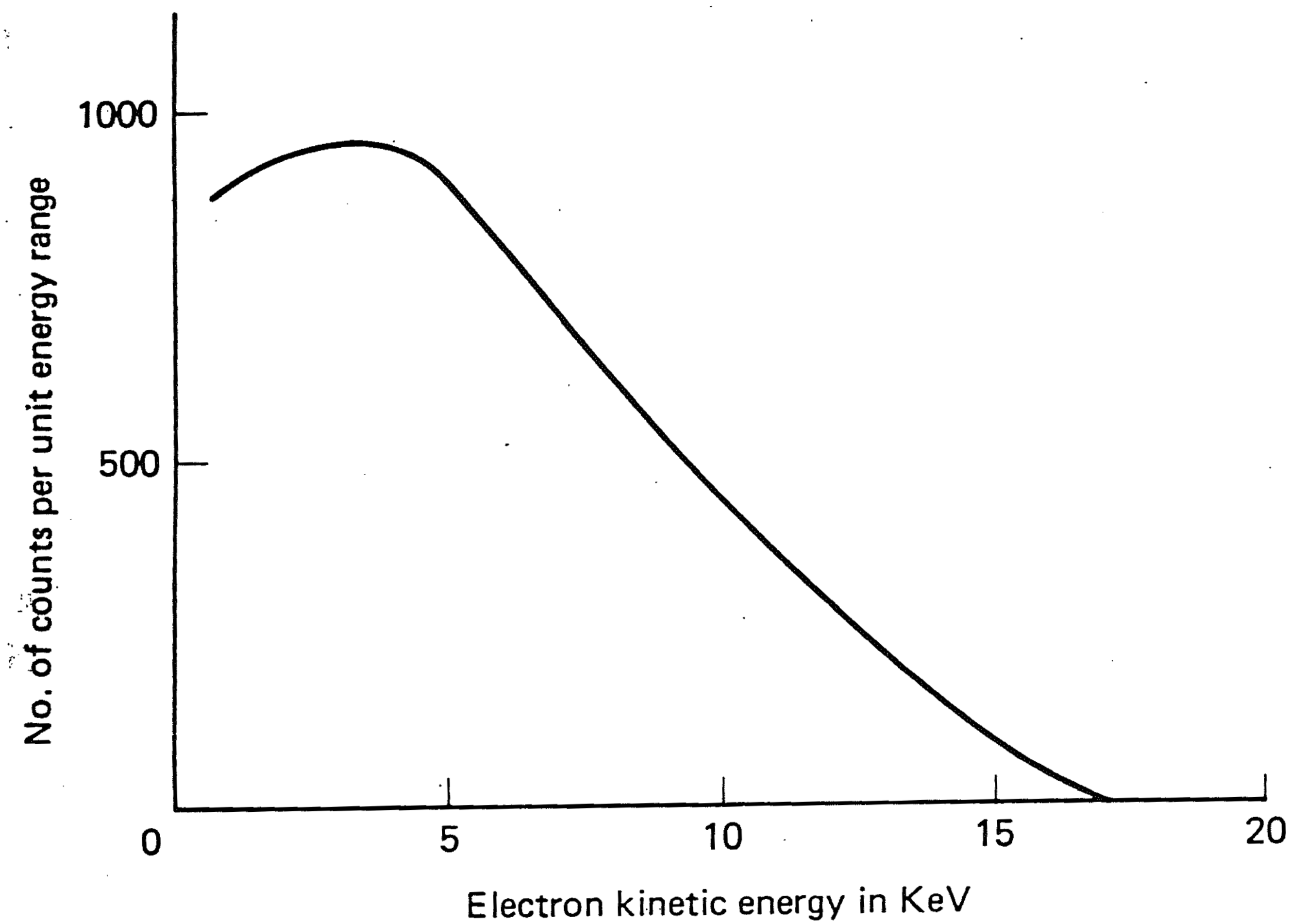
Since $m_e \ll m_H$, $T_{He} \ll T_e$ so expect $T_e \approx Q$

Experiments, however, revealed a

<u>continuous beta spectrum</u> [Canvas]

i.e. T_e can take on any value $0 < T_e < Q$

[Chadwick 1914] see graph $\rightarrow$



The beta decay spectrum of tritium ($^3_1H \rightarrow {}^3_2He$). (*Source:* G. M. Lewis,

Fig. 1.6 Griffiths

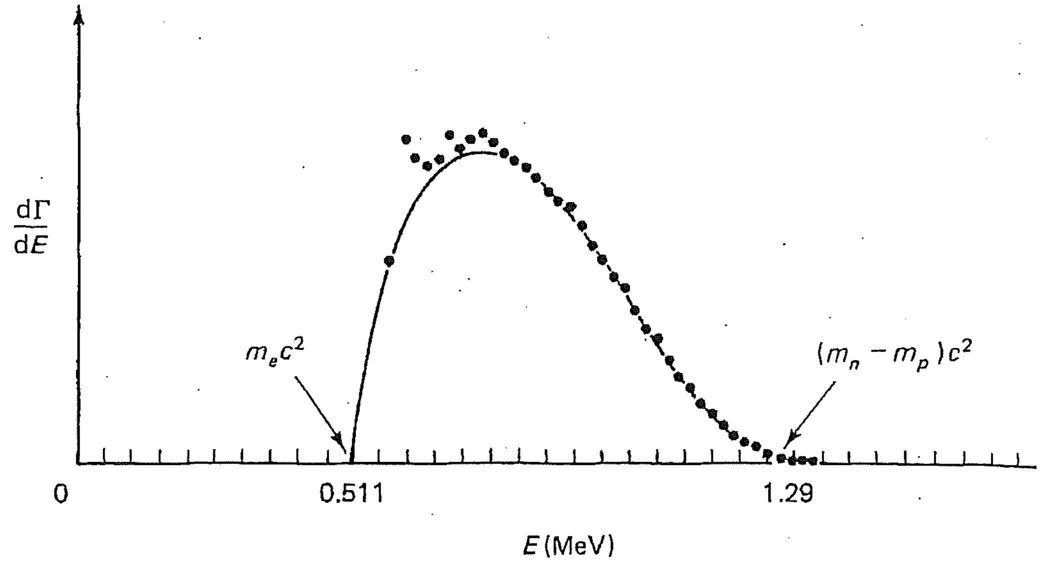


Fig. 9.2 Electron energy distribution from neutron beta decay. (Solid line is the theoretical curve; dots are experimental data.) (*Source*: Christensen, C. J. *et al.* (1972) *Physical Review*, **D5**, 1628. Figure (9.4).)

[Bohr suggested energy not conserved microscopically but only statistically on macro scale.]

1930 Pauli proposed the existence of a neutral particle that carries off the missing energy (w/ $m < 10$ MeV)
Fermi dubbed it the neutrino (little neutral one)

[quote]

$${}^3_1H \longrightarrow {}^3_2He + e^- + \bar{\nu}_e$$

electron antineutrino

As before, T_{He} is negligible so

$$T_e + E_\nu = Q = 18.6 \text{ keV}$$

[NB not T_ν because didn't include m_ν in definition of Q]

$$(E_\nu)_{min} = m_\nu c^2 \quad \text{so}$$

$$(T_e)_{max} = Q - m_\nu c^2$$

Since $(T_e)_{max} \gtrsim 18$ keV $\Rightarrow m_\nu \lesssim 500$ eV [Segre, p. 334]

At any rate the earliest reference I know to the new particle is Heisenberg's mention of 'your neutrons' in a letter to Pauli[97] dated 1 December. More details are found in Pauli's letter (its main part follows) of 4 December to a gathering of experts on radioactivity in Tübingen.[60] 1930

Dear radioactive ladies and gentlemen,
I have come upon a desperate way out regarding the 'wrong' statistics of the N- and the Li 6-nuclei, as well as to the continuous β-spectrum, in order to save the 'alternation law' of statistics* and the energy law. To wit, the possibility that there could exist in the nucleus electrically neutral particles, which I shall call neutrons, which have spin 1/2 and satisfy the exclusion principle and which are further distinct from light-quanta in that they do not move with light velocity. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any case not larger than 0.01 times the proton mass.—The continuous β-spectrum would then become understandable from the assumption that in β-decay a neutron is emitted along with the electron, in such a way that the sum of the energies of the neutron and the electron is constant.

There is the further question, which forces act on the neutron? On wave mechanical grounds . . . the most probable model for the neutron seems to me to be that the neutron at rest is a magnetic dipole with a certain moment μ. Experiments seem to demand that the ionizing action of such a neutron cannot be bigger than that of a γ-ray, and so μ may not be larger than $e \times 10^{-13}$ cm.

For the time being I dare not publish anything about this idea and address myself confidentially first to you, dear radioactive ones, with the question how it would be with the experimental proof of such a neutron, if it were to have a penetrating power equal to or about ten times larger than a γ-ray.

I admit that my way out may not seem very probable *a priori* since one would probably have seen the neutrons a long time ago if they exist. But only he who dares wins, and the seriousness of the situation concerning the continuous β-spectrum is illuminated by my honored predecessor, Mr. Debye, who recently said to me in Brussels: 'Oh, it is best not to think about this at all, as with new taxes'. One must therefore discuss seriously every road to salvation.—Thus, dear radioactive ones, examine and judge.—Unfortunately I cannot appear personally in Tübingen since a ball** which takes place in Zürich the night of the sixth to the seventh of December makes my presence here indispensable . . . Your most humble servant, W. Pauli'.

39. **23.09. 19.08. Since 2007.** *Neutrino mass bound from SN 1987.*
The values of neutrino masses is an open problem in particle physics, but upper bounds (in the range of several eV) have been obtained from tritium endpoint spectra and also from measurements of the cosmic microwave background anisotropy.

An independent bound may be obtained from a burst of neutrinos (and antineutrinos) detected at neutrino observatories in Japan and Cleveland in 1987. These neutrinos (about 24 were detected) were emitted from supernova 1987A, in the large Magellanic cloud, about 50 kpc (or 160,000 light-years) from earth, with a range of energies from about 10 to 40 MeV, with the more energetic neutrinos generally arriving at the earth somewhat earlier. This would be expected if the neutrino has a small mass (much smaller than its energy), as the more energetic neutrinos would have a *slightly* greater speed.

(a) Show that

$$\frac{1}{v} = \frac{1}{c}\left(1 - \frac{1}{\gamma^2}\right)^{-1/2}$$

and then do a Taylor expansion in $1/\gamma$, which is small for highly relativistic neutrinos $v \approx c$.

(b) Consider two neutrinos with the same mass m but different energies E_1 and E_2, and travelling in the same direction. For the purposes of this problem, assume that the neutrinos were emitted at precisely the same instant. Using your result in part (a), estimate the difference in their arrival times a distance L away as a function of their mass and energies.

(c) If the most energetic neutrinos arrive 12 seconds earlier than the least energetic ones, estimate the mass (i.e., rest energy) of the neutrino.
[Note: we now know the neutrino mass to be significantly less than this value, and the likely explanation for the spread in arrival times is simply that the neutrinos were not emitted simultaneously.]

40. *Replaced by problem above. 2007-2023. Neutrino mass bound from SN 1987.*
In 1987, a burst of neutrinos (and antineutrinos) was detected at several neutrino observatories, having been emitted from supernova 1987A, in the large Magellanic cloud, about 170,000 light-years from earth. The neutrinos (about 24 in all) had a range of energies from about 10 to 40 MeV, with the more energetic neutrinos generally arriving at the earth somewhat earlier. This would be expected if the neutrino has a small mass (much smaller than its energy), as the more energetic neutrinos would have a *slightly* greater speed.

(a) Show that for particles moving very close to the speed of light, one may approximate $c - v \simeq c/2\gamma^2$.

(b) Consider two neutrinos with the same mass m but different energies E_1 and E_2, and travelling in the same direction. Using the result from part (a), express the difference Δv in their speeds in terms of m, E_1, and E_2.

(c) For the purposes of this problem, assume that the neutrinos were emitted at precisely the same instant. How does the difference in arrival times depend on the difference on speeds (where the speed is approximately c)?

(d) If the most energetic neutrinos arrive 12 seconds earlier than the least energetic ones, estimate the mass (i.e., rest energy) of the neutrino. [Note: we now know the neutrino

New solution

Upper bound on mass of neutrinos from SN 1987a

$L = 50$ kpc
$= 1.7E5$ ly
$= 1.6$ E21 m

[Griffiths 2e, prob 11.5]

$$v = c\sqrt{1-\frac{1}{\gamma^2}}$$

$$\frac{1}{v} = \frac{1}{c}\left(1-\frac{1}{\gamma^2}\right)^{-1/2}$$

$$= \frac{1}{c}\left(1+\frac{1}{2\gamma^2}+\dots\right)$$

Let Δt be difference in arrival times of neutrinos of energies E_1 and E_2

$$\Delta t = \frac{L}{v_1} - \frac{L}{v_2}$$

$$= \frac{L}{2c}\left(\frac{1}{\gamma_1^2}-\frac{1}{\gamma_2^2}\right)$$

$$= \frac{L}{2c}\left[\frac{(mc^2)^2}{E_1^2} - \frac{(mc^2)^2}{E_2^2}\right]$$

$$\Delta t = \frac{L\,(mc^2)^2}{2c}\,\frac{E_2^2-E_1^2}{E_1^2E_2^2}$$

$$(mc^2)^2 = \frac{2c\,\Delta t}{L}\left(\frac{E_1^2E_2^2}{E_2^2-E_1^2}\right)$$

$E_1 = 10$ meV, $E_2 = 20$ meV, $\Delta t = 12$ s, $L = 1.6$ E21 m

$$(mc^2)^2 = 2(2.2E-12)\left(\underbrace{\frac{(10)^2(40)^2}{(40)^2-(10)^2}}_{107}\ \text{meV}^2\right) = 4.8E-10\ \text{meV}^2$$

$$\boxed{mc^2 = 22\ \text{eV}}$$

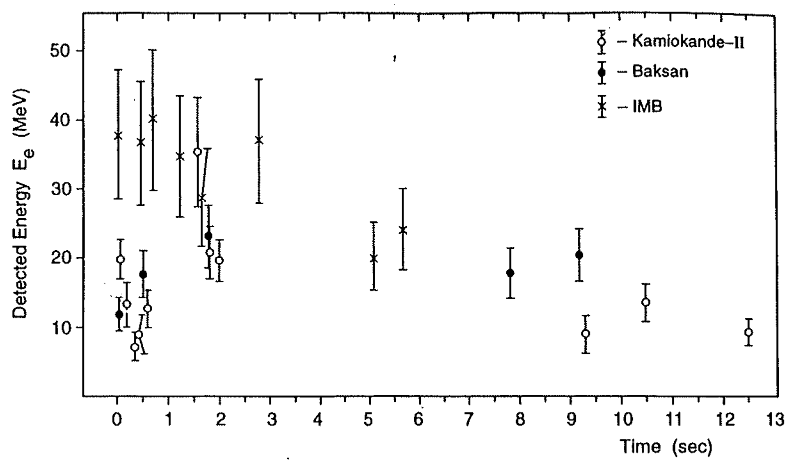
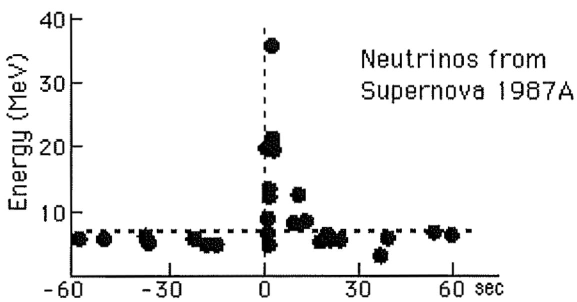


Figure 1: Neutrino light curve for Supernova 1987A. This graph includes *all* neutrinos ever observed from a supernova! (Figure from H. V. Klapdor-Kleingrothaus and K Zuber, *Particle Astrophysics*.)

georgia st


Neutrinos from
Supernova 1987A
Energy (MeV)
40
30
20
10
-60
-30
0
30
60
sec