Liquid drop model of nucleus

Semi-empirical binding energy formula
(Bethe - Weizsacker)

(Fits data well for $A \geq 15$)

$$B = c_1 A - c_2 A^{2/3} - c_3 \frac{Z^2}{A^{1/3}} - c_4 \frac{(A-2Z)^2}{A} + \delta_{pair}$$

A fit to the empirical data gives

[Krane]

$$\begin{cases} c_1 = 15.5 \text{ meV} \\ c_2 = 16.8 \text{ meV} \\ c_3 = 0.72 \text{ meV} \\ c_4 = 23 \text{ meV} \\ c_5 = 34 \text{ meV} \end{cases}$$

$$\delta_{pair} = \begin{cases} c_5 A^{-3/4} & Z, N \text{ even} \\ 0 & Z+N \text{ odd} \\ -c_5 A^{-3/4} & Z, N \text{ odd} \end{cases}$$

c_1 due to strong interaction (volume term) and kinetic energy)

c_2 due to strong interaction (surface term)

c_3 due to EM int. (Coulomb repulsion) $\Rightarrow$ acts to reduce Z

c_4 due to kinetic energy & strong-force $\Rightarrow$ acts to keep $Z \approx N$

c_5 due to spin pairing $\Rightarrow$ favors even even nuclei

$$B = \sum |V| - \sum T$$

Recall

[First think about potential energies due to forces

[Force primarily responsible for existence of nucleus is

strong nuclear force

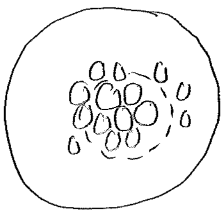
- acts between pairs of nucleons (pp, pn, nn) → think about "bonds" between each pair

  \# bonds: $\frac{A(A-1)}{2}$

  ↓

  Binding energy $\sim \frac{A^2}{2}$ (energy per bond)

but short range, range ~~energy~~ of nucleus R



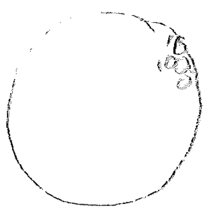
Suppose each nucleon acts on n others for some $n \ll A$

total # of pairs ("bonds") is $\frac{1}{2} n A$

Potential energy = −(# pairs)(energy of "bond")

$$\Rightarrow B \approx c_1 A \quad \text{(volume term)}$$

Then $\frac{B}{A} \approx c_1$ explains why binding energy per nucleon is $\simeq$ const (due to short range nature of force, else $B \sim A^2$)



Nucleons on surface form fewer "bonds"

"missing bonds" ~ # of nucleons on surface $\sim 4\pi R^2 \sim A^{2/3}$

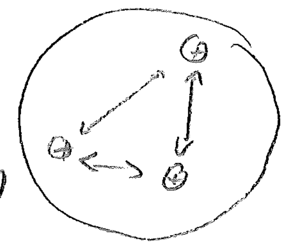
$$B \approx c_1 A - c_2 A^{2/3} \quad \text{(surface term)}$$

Surface term causes nucleus to minimize surface area → sphere

Liquid drop model of nucleus [Bohr]

Electromagnetic potential energy

- acts between pairs of protons: $\frac{Z(Z-1)}{2}$ pairs
- long range: affects all protons in nucleus
- repulsive $\Rightarrow$ $V > 0$ $\Rightarrow$ reduces binding energy



Treat protons as uniformly distributed charge $q = Ze$

Potential energy of sphere of charge $V = \frac{3}{5}\frac{Kq^2}{R} = \underbrace{\left(\frac{3Ke^2}{5r_0}\right)}_{c_3}\frac{Z^2}{A^{1/3}}$

Estimate $c_3 = \frac{3Ke^2}{5r_0} = \frac{(0.6)(1.44\ \text{MeV}\cdot\text{fm})}{(1.2\ \text{fm})} = \underline{\underline{0.72\ \text{MeV}}}$

[Krane, p. 67]

$$B \approx c_1 A - c_2 A^{2/3} - c_3 \frac{Z^2}{A^{1/3}} - c_4 \frac{(A-2Z)^2}{A}$$

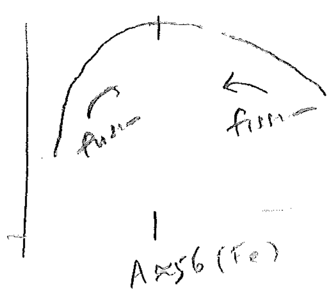
$$\frac{B}{A} \approx c_1 - \frac{c_2}{A^{1/3}} - c_3 \frac{Z^2}{A^{4/3}} - c_4 \left(\frac{A-2Z}{A}\right)^2$$

causes curve to bend down for small A

[hydrogen has B = 0]

If assume $Z = \frac{A}{2}$, $\frac{B}{A} = c_1 - \frac{c_2}{A^{1/3}} - \frac{c_3}{4} A^{2/3}$

causes curve to bend down for large Z & A & eventually nucleus becomes unstable



[EM acts on all protons, strong force only on nearby nucleons]

Recall $B = (\Sigma|V|) - (\Sigma T)$

[B depends not only on potential energies
but also on kinetic energies of the nucleons
(not the nucleus itself, which is at rest)

But aren't the nucleons at rest?

In a gas of particles, the kinetic energy depends on temperature
but at zero temperature, expect kinetic energy to vanish

True classically but not quantum mechanically,
due to Heisenberg uncertainty principle

Particles in a confined space ($\Delta x < L$) have $\Delta p \sim \frac{\hbar}{L}$

i.e. nonzero momentum,

and therefore nonzero kinetic energy.

Also Pauli exclusion principle means different nucleons
have different momenta &
So the more nucleons, the higher the allowed
momenta & ∴ higher the kinetic energy]

Heisenberg uncertainty + Pauli exclusion $\Rightarrow$ nonzero kinetic energy of nucleons

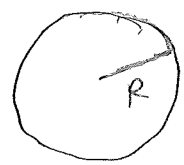
[First consider a single nucleon]

Describe nucleon w/ wavefunction ψ

[satisfies Schrodinger eqn in some potential well,
complicated interactions w/ other nucleons but
assume they all average out,
except that they keep nucleon inside the nucleus]

particle in a (spherical) box

$$R = A^{\frac{1}{3}} r_0$$



[Hard to solve Schrodinger eqn in spherical coords]

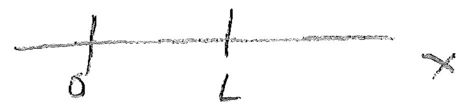
Replace sphere w/ cube of same volume

L

$$V = L^3 = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi A r_0^3$$

$$L = r_0 \sqrt{\frac{4\pi A}{3}}$$

First consider 1d box

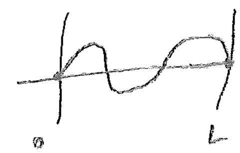


Boundary conditions

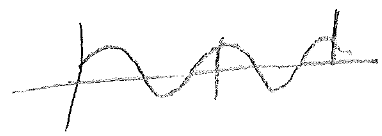
Dirichlet: ψ vanishes at boundary

$$\psi = \sin\left(\frac{\pi n x}{L}\right) \qquad n \in \mathbb{Z}$$

[standing wave]



periodic: $\psi(x+L) = \psi(x)$



free particle: $\psi = e^{\frac{ipx}{\hbar}}$

[travelling wave]

$$e^{\frac{ip(x+L)}{\hbar}} = e^{\frac{ipx}{\hbar}}$$

$$e^{\frac{ipL}{\hbar}} = 1$$

$$\frac{pL}{\hbar} = 2\pi n \qquad (n \in \mathbb{Z})$$

$$p = \frac{2\pi\hbar}{L} n = \frac{h}{L} n$$

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3 dim's

$$\psi = e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}} = e^{\frac{ip_x x}{\hbar}} e^{\frac{ip_y y}{\hbar}} e^{\frac{ip_z z}{\hbar}}$$

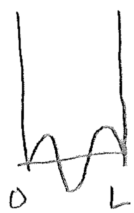
$$\vec{p} = (p_x, p_y, p_z) = \frac{h}{L}(n_x, n_y, n_z) = \frac{h}{L}\vec{n}$$

momenta are quantized

periodic vs Dirichlet

Dirichlet vs periodic b.c.

1. Consider a 1-dimensional box of length L.



If we impose Dirichlet b.c. on the particle (i.e. $\psi(x)$ vanishes at edges of box)

then de Broglie wavelength $\lambda = \frac{2L}{n}$

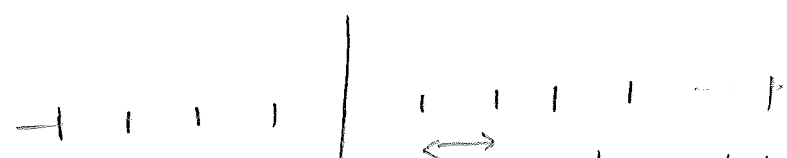
then $|\vec{p}| = \frac{h}{\lambda} = \frac{hn}{2L}$, $n = 1, 2, 3, \ldots$

momentum spacing is $\frac{h}{2L}$

Alternatively one can impose periodic b.c. $\psi(x+L) = \psi(x)$

Then $\psi = e^{\frac{ipx}{\hbar}} \Rightarrow e^{\frac{ipL}{\hbar}} = 1 \Rightarrow \frac{pL}{\hbar} = 2\pi n$

so $p = \frac{hn}{L}$, $n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

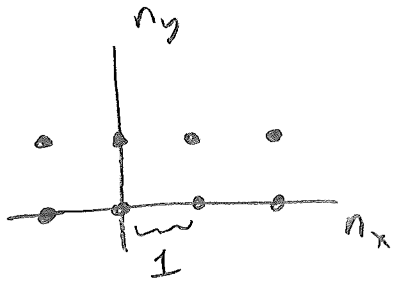


spacing is $\frac{h}{L}$ but negative p are allowed

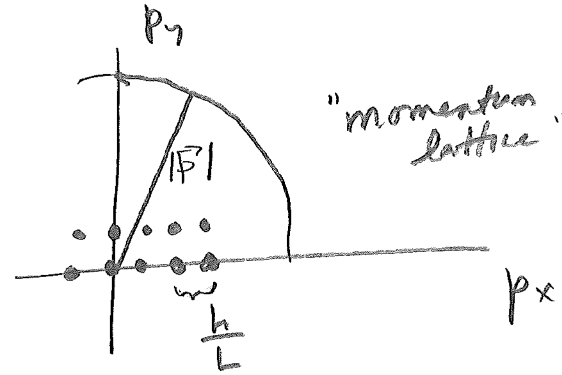
These give equivalent results because both give same # of states for $|\vec{p}| \leq p_{max}$,

namely $N_{states} = \frac{p_{max}}{\left(\frac{h}{2L}\right)} = \frac{2\,p_{max}}{\left(\frac{h}{L}\right)}$

Allowed states of a single nucleon are described by points on a 3d lattice



or



Kinetic energy of a nucleon increases w/ momentum $|\vec{p}|$

e.g. Nonrelativistic $\quad T = \frac{|\vec{p}|^2}{2m} = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right)$

Rel. $T = \sqrt{(c\vec{p})^2 + (mc^2)^2} - mc^2$

ultraRelativistic $\quad T \approx c|\vec{p}| = c\sqrt{p_x^2 + p_y^2 + p_z^2}$

Number of allowed states within a given radius n is given by counting lattice points

$$\sum_{\vec{n}} 1 = \underbrace{\sum_{n_x}\sum_{n_y}\sum_{n_z}}_{\substack{\text{subject to} \\ n_x^2+n_y^2+n_z^2 \le n}} 1 \approx \int dn_x\, dn_y\, dn_z \; 1 = \text{volume of sphere of radius } n$$

because there is one lattice point per unit volume

[Not exact, but accurate for large n]

Recall nucleons (p, n) are spin-½ particles and therefore fermions.

They have two distinct spin states: ↑ and ↓

The quantum state of a nucleon is determined by its momentum and its spin direction

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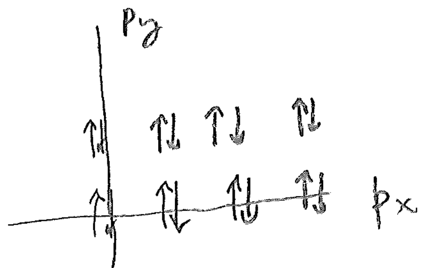
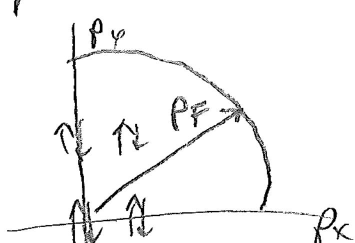
<u>Pauli exclusion principle</u>: no two <u>identical</u> fermions can occupy the same quantum state

⇒ a collection of identical fermions must occupy states w/ distinct momentum & spin



At zero temperature, N neutrons will occupy the N quantum states w/ lowest kinetic energy (∴ smallest momentum) up to some maximum momentum p_F (Fermi momentum)



"degenerate Fermi gas"

[How do we compute p_F?]

$$N = \sum_{\vec{n}} 2 \approx \int dn_x\, dn_y\, dn_z \cdot 2$$

lattice points w/ $|p| \leq p_F$ — spin multiplicity

Now $\vec{n} = \frac{L}{h}\vec{p}$ so $dn_x = \frac{L}{h} dp_x$ etc

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$$N \approx \frac{L^3}{h^3} \int dp_x\, dp_y\, dp_z \cdot 2$$

$$= \frac{2L^3}{h^3} \int d^3p$$

$$= \frac{2L^3}{h^3} \int_0^{p_F} 4\pi p^2\, dp$$

$$= \frac{2L^3}{h^3} \left(\frac{4\pi p_F^3}{3} \right)$$

$$N = \frac{8\pi}{3} \left(\frac{L p_F}{h} \right)^3$$

$$\Rightarrow p_F = \frac{h}{L} \left(\frac{3N}{8\pi} \right)^{1/3}$$ neutron Fermi momentum

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Z protons constitute a separate degenerate Fermi gas w/ their own Fermi momentum

$$p_F = \frac{h}{L} \left(\frac{3Z}{8\pi} \right)^{\frac{1}{3}}$$

What is the kinetic energy of this degenerate fermi gas?

$$T^{tot} = 2 \sum_{\vec{n}} T_{\vec{n}} \approx 2 \int d^3n \, T_n = \frac{2L^3}{h^3} \int d^3p \, T(p)$$

$$= \frac{2L^3}{h^3} \int (4\pi p^2 dp) \, T(p)$$

(spin up & down have same kinetic energy)

Recall: For nonrelativistic particles ($v \ll c$), $T \approx \frac{p^2}{2m}$

For ultrarelativistic particles ($v \approx c$), $T \approx cp$

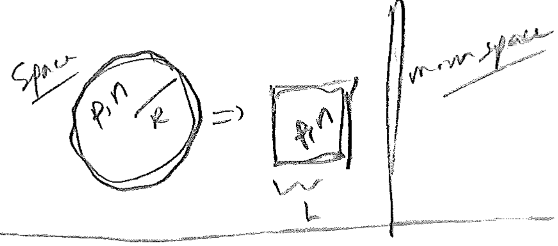
[HW] Show that average kinetic energy $\bar{T} = \frac{3}{5} \epsilon_F$

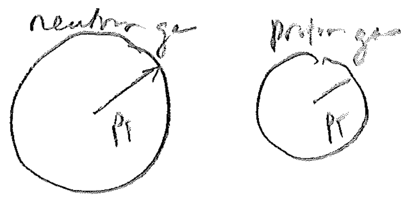
where $\epsilon_F = \frac{p_F^2}{2m}$ is the nonrelativistic fermi energy

The total kinetic energy of a degenerate gas of N nonrelativistic neutrons is then

$$T^{tot} = N \bar{T} = \frac{3}{10} \frac{p_F^2}{m} N$$

[HW] Calculate the average kinetic energy of a degenerate gas of ultrarelativistic particles





Recall fermi momentum of a degenerate gas of N neutrons

$$p_F = \frac{h}{L}\left(\frac{3N}{8\pi}\right)^{\frac{1}{3}} = \hbar\left(\frac{3\pi^2 N}{L^3}\right)^{\frac{1}{3}}$$

Replace cube volume L^3 by sphere volume $\frac{4\pi}{3} A r_0^3$

$$p_F = \frac{\hbar}{r_0}\left(\frac{9\pi}{4}\frac{N}{A}\right)^{\frac{1}{3}}$$

Degenerate gas is <u>nonrelativistic</u> if $\frac{v}{c} \ll 1 \Rightarrow \frac{p}{mc} \ll 1$

The max momentum of the gas is p_F

$$\frac{p_F}{mc} = \frac{\hbar c}{mc^2 r_0}\underbrace{\left(\frac{9\pi}{4}\frac{N}{A}\right)^{\frac{1}{3}}}_{\text{of order } 1} \sim \frac{(200\ \text{MeV}\cdot\text{fm})}{(940\ \text{MeV})(1.2\ \text{fm})} \sim \frac{1}{5} \checkmark$$

we can use nonrelativistic kinetic energy

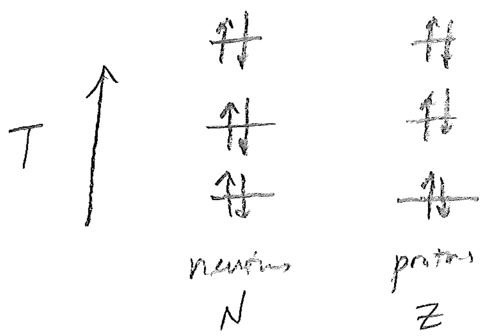
$$\Rightarrow T^{tot} = \frac{3}{10}\frac{p_F^2}{m}\frac{N}{A}\cdot A = \underbrace{\frac{3}{10}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}}_{\sim 1.105}\frac{\hbar^2}{m r_0^2}\left(\frac{N}{A}\right)^{\frac{5}{3}}\cdot A$$

Total kinetic energy of neutron and proton gas is

$$T = 1.105\frac{\hbar^2}{m r_0^2}\left[\left(\frac{N}{A}\right)^{\frac{5}{3}} + \left(\frac{Z}{A}\right)^{\frac{5}{3}}\right] A$$

(masses about the same)

First, consider nuclei w/ $Z \simeq N \simeq \frac{1}{2}A$



$$\frac{(\hbar c)^2}{(mc^2)\, r_0^2} = \frac{(197)^2}{(940)(1.2)^2} = 29 \text{ MeV}$$

$$T^{tot} = \underbrace{1.105 \left[\left(\tfrac{1}{2}\right)^{5/3} + \left(\tfrac{1}{2}\right)^{5/3} \right]}_{\simeq\, 0.696} \underbrace{\frac{\hbar^2}{m r_0^2}}_{29 \text{ MeV}} A$$

$\approx (20 \text{ MeV})\, A$ ← reduces binding energy

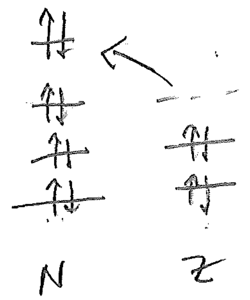
[about what we found for deuteron]

[This is a contribution to the binding energy <u>volume term</u>

Suggests that potential energy per nucleon is ~ -35 MeV

$B = -T^{tot} - V^{tot} \approx (15 \text{ MeV})\, A$]

If $N \neq Z$, kinetic energy increases. Why?

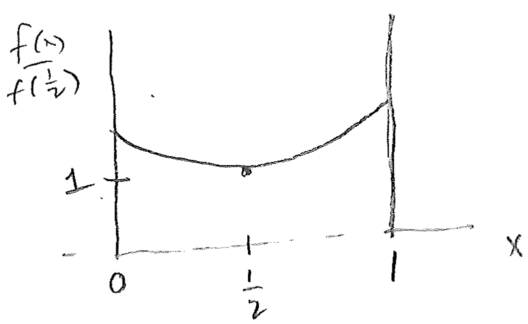


Let's estimate change in T^{tot}.

Let $x = \frac{Z}{A}$

Then $T^{tot} = (20\ \text{MeV})\, A \cdot \frac{f(x)}{f(\frac{1}{2})}$

where $f(x) = x^{5/3} + (1-x)^{5/3}$ $\qquad f(\frac{1}{2}) \simeq 0.63$



Near $x = \frac{1}{2}$, curve is $\approx$ parabolic

$$\frac{f(x)}{f(\frac{1}{2})} = 1 + \#\left(x - \tfrac{1}{2}\right)^2 + \ldots \qquad \left[\# = \tfrac{20}{9}\right]$$

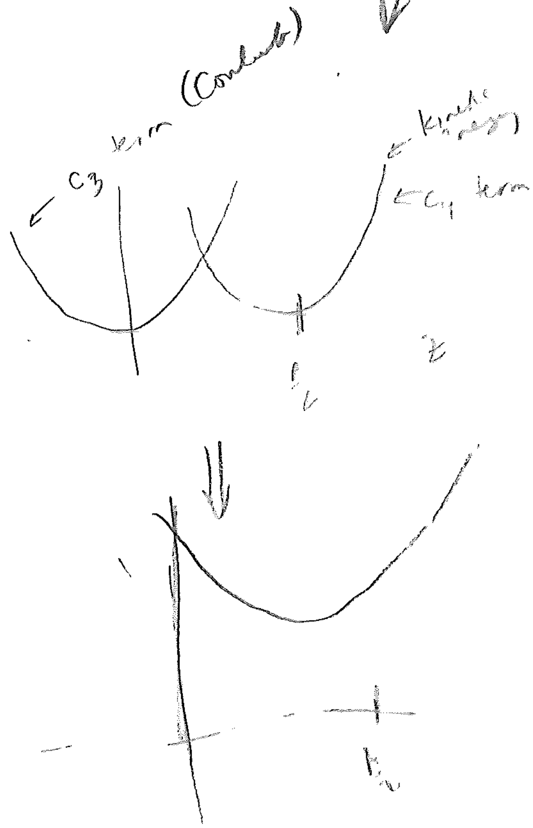
HW: calculate # + use it estimate c_4 in semi-empirical

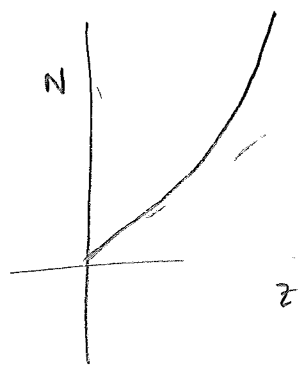
Hint: it doesn't agree too well, but is right order of mag

[$c_4 = 11$ MeV, whereas actual $c_4 = 23$ MeV]

$$B = c_1 A - c_2 A^{2/3} - c_3 \underbrace{\frac{Z^2}{A^{1/3}}} - c_4 \underbrace{\frac{(A-2Z)^2}{A}} + \delta_{pair}$$

favors $Z \ll N$	favors $Z = \frac{A}{2}$ or $Z = N$
more effective at large A	more effective at small A
$\Downarrow$	$\Downarrow$
heavy nuclei have fewer protons than neutrons	light nuclei have $Z \approx N$ — ^4He, ^{12}C, ^{16}O





[problem: solve for Z as a fcn of A that minimizes Δ (not B)]

Calc of asymmetry term in semi-empirical formula

Kinetic energy of nucleus

$$\vec{k} = \frac{2\pi}{L}\vec{n}$$

$$\frac{4\pi}{3} n_{max}^3 = \frac{N}{2}$$

$$\epsilon_F = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2\left(\frac{3N}{8\pi}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$U = \frac{3}{5} N \epsilon_F$$

$r_0 = 1.2$ fm (Serway)
$ = 1.5$ fm (Fermi)

For a nucleus. $V = A V_0$, $V_0 = \frac{4}{3}\pi r_0^3$

$$\epsilon_F = \underbrace{\frac{\hbar^2 c^2}{2mc^2 r_0^2}\left(\frac{9\pi}{4}\right)^{2/3}}_{C_0} \left(\frac{N}{A}\right)^{2/3}$$

53 MeV (Serway)
34 MeV (Fermi)

$\rightarrow N_p = N_n = \frac{A}{2} \Rightarrow \epsilon_F \approx 21$ MeV/fe

if only neutrons, $\epsilon_F = \begin{Bmatrix} 53 \\ 34 \end{Bmatrix}$

27 MeV
say [illegible]
$r_0 = 1.3$ f

$$U_{Nuc} = \frac{3}{5} N_p C_0 \left(\frac{N_p}{A}\right)^{2/3} + \frac{3}{5} N_n C_0 \left(\frac{N_n}{A}\right)^{2/3}$$

$$= \frac{3}{5}\frac{C_0}{A^{2/3}}\left[\underbrace{Z^{5/3} + (A-Z)^{5/3}}_{f(Z,A)}\right]$$

$$\rho = Z - \frac{A}{2}$$

$$f(Z,A) = \left(\frac{A}{2}+\rho\right)^{5/3} + \left(\frac{A}{2}-\rho\right)^{5/3}$$

$$= \frac{A^{5/3}}{2^{3/2}}\left[1 + \frac{20}{9}\frac{\rho^2}{A^2}\right] = (.63)A^{5/3} + (.35)A^{-1/3}(A-2Z)^2$$

$$U_{Nuc} = \underbrace{(.38)\,C_0}_{\substack{20\text{ MeV (Serway)} \\ 13\text{ MeV (Fermi)}}} A + \underbrace{(.21)\,C_0}_{\substack{11\text{ MeV (Serway)} \\ 7\text{ MeV (Fermi)}}} \frac{(A-2Z)^2}{A}$$

If all neutrons, $(A = N_n, Z = 0)$ $U_{Nuc} = \frac{3}{5} C_0 A = \begin{Bmatrix} 32\text{ MeV} \\ 20\text{ MeV} \end{Bmatrix} A$,